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SYMMETRIC HOLOMORPHICALLY SUBPROJECTIVE KAHLERIAN MANIFOLD

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Two tensors U_{ij} and V_{ij} are introduced to find a necessary condition, in terms of these tensors, for a holomorphically subprojective Kahlerian manifold to be symmetric space. Further, it is pointed out that the conditions, namely $1 + \rho_r x^r = 0$ and $x_r \rho^r = 0$ used by *T.* Adati[5] in proving a theorem, can be replaced by the one involving U_{ij}.

1. PRELIMINARIES

Let (M_{2n}, F_j^i, g_{ij}) be a complex structure on $2n$-real dimensional Kahlerian manifold M_{2n}. Then

$$F_j^i F_i^h = -\delta_j^h, \; F_j^t F_i^s g_{ts} = g_{ji}, \; F_j^t g_{ti} = F_{ji}$$

and

$$\nabla_j F_i^h = 0 \text{ (hence } \nabla_j F_{ih} = 0). \qquad \ldots (1.1)$$

A tensorfield with components T_{ijk} .... is called recurrent in a manifold if[2]

$$\nabla_1 T_{ijk} \ldots\ldots = k_1 T_{ijk} \ldots\ldots,$$

where the vector field $k_1 \neq 0$ is called recurrent vector field. If $k_1 = 0$, then the manifold is called T-symmetric[1]. In particular if the curvature tensorfield R_{kjih} is recurrent, then the manifold is called recurrent manifold, and it is called symmetric if R_{kjih} is symmetric.

Yamaguchi and Adati[4, 5] have introduced and studied holomorphically subprojective Kahlerian manifold similar to the subprojective real manifold introduced by Kagan[3]. They have proved that the necessary and sufficient condition for $2n$ $(n \geq 3)$ real dimensional Kahlerian manifold M_{2n} to be holomorphically subprojective is that there exists a local real co-ordinate system (x^h) such that Christoffel symbol $\left\{ {h \atop ij} \right\}$ of M_{2n} is of the form:

$$\left\{ {h \atop ij} \right\} = \rho \; {}_{(i}\delta_{j)}^h + \tilde{\rho} \; {}_{(i}F_{j)}^h + f_{ji} x^h - f_{jr} F_1^r \tilde{x}^h,$$

with

$$f_{[ij]} = 0 \text{ and } f_r\,[{}_j F^r_i] = 0,$$

where ρ_i and f_{ji} are covariant vector and tensorfields respectively,

$\tilde{x}^h = F^h_r\, x^r$, $\tilde{\rho}_h = -F^r_h\, \rho_r$ and (i, j) (respectively $[i, j]$) represent symmetric (respectively skew symmetric) with respect to the indices i and j. Thus on a holomorphically sub projective Kahlerian manifold, we have

$$\text{(i) } f_{ij} = f_{ji}, f_{rj}\, F^r_i = f_{ri}\, F^r_j \qquad \text{... (1.2)}$$

and

$$\text{(ii) } \tilde{x}^h = F^h_r\, x^r, \tilde{\rho}_h = -F^r_h\, \rho_r, x^t = F^t_h\, x^h, \rho_t = F^h_t\, \tilde{\rho}_h.$$

Further, the components R_{kjih} of curvature tensor in holomorphically subprojective Kahlerian manifold are given by[5]

$$R_{kjih} = \left\{ a + \frac{\lambda + \varepsilon}{2(n-1)} \,|\, x \,|^2 \right\} (g_i\,[{}_k g_j]h - F_i\,[{}_k F_j]h + 2F_{kj}\, F_{ih})$$

$$- \frac{(\lambda + \varepsilon)}{2(n-1)} \Big[(F_i[{}_k \tilde{x}_j] + 2F_{jk}\, \tilde{x}_i - g_i[{}_j x_k])\, x_h) - (F_i[{}_k x_j] + 2F_{jk}\, x_i + g_i[{}_j \tilde{x}_k])\, \tilde{x}_h$$

$$+\; x_i\, (x_k[g_j]h - \tilde{x}_k F_j]h) + \tilde{x}_i\, (\tilde{x}_k[g_j]h + x_k\, [F_j]h) - 2F_{ih}\, \tilde{x}_k[x_j] \Big]$$

$$- \frac{2\lambda + (n+1)\varepsilon}{(n-1)\,|\, x \,|^2}\, x_{[{}_k \tilde{x}_j]}\, \tilde{x}_{[{}_i x_h]},$$

where a, λ and ε are constants and

$$|\, x \,|^2 = g_{ij}\, x^i\, x^j \qquad \text{... (1.4)}$$

Yamaguchi and Adati have also obtained

$$\nabla_j x_i = (1 + \rho_r x^r)\, g_{ji} - \rho_r \tilde{x}^r\, F_{ji} + (\rho_j + f_{jr}\, x^r)\, x_i + (\tilde{\rho}_j + f_{jr}\, \tilde{x}^r)\, \tilde{x}_i \qquad \text{... (1.5)}$$

$$\nabla_j\, \tilde{x}_i = (1 + \rho_r x^r)\, F_{ji} + \rho_r \tilde{x}^r\, g_{ji} + (\rho_j + f_{jr}\, x^r)\, x_i - (\tilde{\rho}_j - f_{jr}\, \tilde{x}^r)\, \tilde{x}_i \qquad \text{... (1.6)}$$

$$\nabla_j\, \rho_i = -\rho_j\, \rho_i + \tilde{\rho}_j\, \rho_i + f_{ji} + \frac{1}{2(n-1)} \left\{ 2(n-1)\, a + (\lambda + \varepsilon)\,|\, x \,|^2 \right\} g_{ji}. \qquad \text{... (1.7)}$$

A holomorphically subprojective Kahlerian manifold is said to be of first kind if[5]

$$2\lambda + (n+1)\,\varepsilon = 0.$$

2. Lemmas and Theorems

Corresponding to the local co-ordinate system (x^h) we define

$$U_{ij} = x_i \tilde{x}_j - \tilde{x}_i x_j \quad \text{... (2.1)}$$

and

$$V_{ij} = x_i x_j + \tilde{x}_i \tilde{x}_j. \quad \text{... (2.2)}$$

Clearly $U_{ij} = -U_{ji}$ and $V_{ij} = V_{ji}$.

Lemma 2.1 — On a holomorphically subprojective Kahlerian manifold M_{2n} with local co-ordinates (x^h), U_{ij} and V_{ij} satisfy the following conditions.

$$(i)\ V_{ij} = F_j^r U_{ir} = -F_i^r U_{ri}, \quad \text{... (1)}$$

$$(ii)\ U_{ij} = F_i^r V_{jr} = F_i^r V_{rj} \quad \text{... (2)}$$

and $$(iii)\ V_{ht} V_l^t = U_{ht} U_l^t = |x|^2 V_{hl}, \text{ where } U_l^t = g^{tj} U_{lj} \text{ and } V_l^t = g^{tj} V_{lj}.$$

PROOF : By using (1.1) and (1.2) we get

$$\tilde{x}_t = -F_t^h x_h \text{ and } x_t = F_t^h \tilde{x}_h. \quad \text{... (2.3)}$$

Now by using (1.2) and (2.3) in (2.2) we get

$$V_{ij} = F_j^r (x_i \tilde{x}_r - \tilde{x}_i x_r)$$

which in view of (2.1) and the fact $U_{ij} = -U_{ji}$ gives (i).

Similarly, by using (2.3) and (1.2) in (2.1) we get (ii).

Since $V_{ij} = V_{ji}$, it follows that

$$F_j^r U_{ri} = F_i^r U_{rj}.$$

Transvecting the above result by U_l^j, we get

$$U_l^j F_j^r U_{ri} = U_l^j F_i^r U_{rj}$$

which by using $F_j^r = -F_j^r$ and $U_{ij} = -U_{ji}$ reduces to the form :

$$U_{jl} F_r^j U_i^r = U_l^j F_i^r U_{rj}.$$

In view of the result (ii), the above equation can be written as

$$V_l^r U_{ri} = -U_t^j F_i^r U_{rj}$$

which after transvecting by F_h^i and using (1.1) gives

$$F_h^i U_{ri} V_l^r = U_l^j U_{hj}$$

and hence

$$V_{ht} V_l^t = U_{ht} U_l^t, \qquad \text{... (2.4)}$$

by virtue of (ii).

Further by substituting (2.1), (2.2) in (2.4), we get

$$(|\tilde{x}|^2 - |x|^2)(\tilde{x}_h \tilde{x}_l - x_l x_h) + 2x_t \tilde{x}^t (x_h \tilde{x}_l - \tilde{x}_h x_l) = 0, \qquad \text{... (2.5)}$$

where

$$|\tilde{x}|^2 = \tilde{x}^t \tilde{x}_t \text{ and } |x|^2 = x^t x_t.$$

But $|\tilde{x}|^2 = \tilde{x}^t \tilde{x}_t = (F_r^t x^r)(-F_t^h x_h) = |x|^2$ by virtue of (2.1) and (1.1).

Hence from (2.5), we obtain

$$x_t \tilde{x}^t (x_h \tilde{x}_l + \tilde{x}_h x_l) = 0 \qquad \text{... (2.6)}$$

which by transvecting with g^{hl} gives

$$x_t \tilde{x}^t = 0. \qquad \text{... (2.7)}$$

Using (2.7) in the expression,

$$V_{kt} V_l^t = (x_k x_t + \tilde{x}_k \tilde{x}_t)(x_l x^t + \tilde{x}_l \tilde{x}^t) \text{ we get (iii).}$$

Lemma 2.2 — Suppose M_{2n} is a holomorphically subprojective Kahlerian manifold corresponding to a local real co-ordinate system (x^h), then

(i) $f_l{}^r U_j^r = 0$ if $f_l{}^r x^r = 0$

and (ii) $x_r \rho^r = \tilde{x}_r \rho^r = x_r \tilde{\rho}^r = \tilde{x}_r \tilde{\rho}^r = 0$, if $U_{ir} \rho^r = 0$.

PROOF : By using $f_{lr} x^r = 0$ and definition of U_{jr}, we get $f_{lr} U_j^r = 0$, which proves (i).

Now suppose $U_{lr} \rho^r = 0$. Then by using the result of (ii) of lemma (2.1), we get

$$F_l^t V_{rt} \rho^r = 0,$$

which, after transvecting with F_k^l and using (1.1), gives $V_{lr}\rho^r = 0$.

Since $U_{lr}\rho^r = V_{lr}\rho^r$, we get

$$x_r \rho^r (\tilde{x}_l + x_l) + \tilde{x}_r \rho^r (\tilde{x}_l - x_l) = 0, \quad \text{... (2.8)}$$

which by transvecting with x^l and using (2.7) along with $|x| \neq 0$ gives

$$x_r \rho^r = \tilde{x}_r \rho^r. \quad \text{... (2.9)}$$

So from (2.8), we get $2\tilde{x}_l x_r \rho^r = 0$, which leads to

$$x_r \rho^r = 0. \quad \text{... (2.10)}$$

Now in view of (2.9), we also have

$$\tilde{x}_r \rho^r = 0. \quad \text{... (2.11)}$$

Further, by using (1.2) in (2.11), we obtain

$$\tilde{\rho}_r x^r = 0 \quad \text{... (2.12)}$$

and similarly we can prove

$$\tilde{\rho}^r \tilde{x}^r = 0. \quad \text{... (2.13)}$$

If we assume $U_{lr}\rho^r = 0$ and $f_{lr}x^r = 0$, then by using the results of Lemma 2.2, we can reduce the results (1.5), (1.6) and (1.7) to the form

$$\nabla_j x_i = g_{ji} + \rho_j x_i + \tilde{\rho}_j \tilde{x}_i, \quad \text{... (2.14)}$$

$$\nabla_j \tilde{x}_i = F_{ji} + \rho_j \tilde{x}_i - \tilde{\rho}_j x_i \quad \text{... (2.15)}$$

and

$$\nabla_j \rho_i = -\rho_j \rho_i + \tilde{\rho}_j \rho_i + f_{ji}$$
$$+ \frac{1}{2(n-1)} \left[\left\{ 2(n-1)\, a + (\lambda + \varepsilon) |x|^2 \right\} g_{ji} - (\lambda + \varepsilon)\, V_{ji} \right]. \quad \text{... (2.16)}$$

Lemma 2.3 — Suppose M_{2n} is holomorphically subprojective Kahlerian manifold with local co-ordinates (x^h), then

$$\nabla_1 U_j^1 = 2n\tilde{x}_j, \text{ if } U_{1i}\rho^i = 0 \text{ and } f_{1r}x^r = 0.$$

PROOF : Operate ∇_1 on both sides of (2.1) and use (2.14) and (2.15) to get

$$\nabla_1 U_{ij} = F_{1j} x_i - F_{1i} x_j + \tilde{x}_j g_{1i} - \tilde{x}_i g_{1j} + 2\rho_1 U_{ij}. \quad \text{... (2.17)}$$

On transvecting this result with g^{il} and using (1.2) we obtain the result.

Theorem 2.1 — *A holomorphically subprojective Kahlerian manifold is symmetric if it admits a co-ordinate system* (x^h) *satisfying* $U_{1r}\rho^r = 0 = f_{1r} x^r$ *and* $\tilde{x}_j \rho_1 + x_j \tilde{\rho}_1 = 0$.

PROOF : By Ricci identity, we have

$$\nabla_k \nabla_l U_{ij} - \nabla_l \nabla_k U_{ij} = -\left[R_{kli}^{\ \ \ h} U_{hj} + R_{klj}^{\ \ \ h} U_{ih} \right].$$

By substituting (2.1) in (1.3) we obtain the expression,

$$\begin{aligned} R_{kjih} = & \left\{ a + \frac{\lambda + \varepsilon}{2(n-1)} |x|^2 \right\} (g_{ki} g_{jh} - g_{ji} g_{kh} + F_{ki} F_{jh} - F_{ji} F_{kh} + 2F_{kj} F_{ih}) \\ & + \frac{(\lambda+\varepsilon)}{2(n-1)} \begin{pmatrix} F_{ji} U_{kh} U_{jh} - F_{jh} U_{ki} + F_{kh} U_{ji} - 2F_{kj} U_{ih} \\ -2F_{ih} U_{kj} + g_{ji} V_{kh} - g_{ki} V_{jh} - g_{jh} V_{ki} + g_{kh} V_{ji} \end{pmatrix} \\ & + \frac{2\lambda + (n+1)\varepsilon}{(n-1)|x|^2} U_{kj} U_{ih}, \end{aligned} \quad \text{... (2.18)}$$

for curvature tensor in terms of U_{ij} and V_{ij} which reduces the above identity to the form

$$\begin{aligned} \nabla_k \nabla_l U_{ij} - \nabla_l \nabla_k U_{ij} = & -\left(a + \frac{\lambda+\varepsilon}{2(n-1)} \right) \begin{pmatrix} g_{ki} U_{lj} - g_{kj} U_{li}) + (g_{lj} U_{ki} - g_{li} U_{kj}) \\ + (F_{li} V_{jk} - F_{lj} V_{ik}) + (F_{kj} V_{li} - F_{ki} V_{jl}) \end{pmatrix} \\ & - \frac{\lambda+\varepsilon}{2(n-1)} \begin{pmatrix} (F_{li} U_{hj} + F_{lj} U_{ih}) U_k^h - (F_{ki} U_{hj} + F_{kj} U_{ih}) U_l^h \\ -2F_{kl} (U_{hj} U_i^h + U_{ih} U_j^h) - (g_{ki} U_{hj}) - g_{kj} U_{ih}) V_l^h \\ + (g_{li} U_{hj} - g_{lk} U_{ij}) V_k^h. \end{pmatrix} \end{aligned}$$

By using Lemma 2.1 we can reduce the above equation to

$$\nabla_k \nabla_1 U_{ij} - \nabla_1 \nabla_k U_{ij} = -a \begin{bmatrix} (g_{ki} U_{1j} - g_{kj} U_{1i}) - (g_{1i} U_{kj} - g_{1j} U_{ki}) \\ -(F_{ki} V_{j1} - F_{kj} V_{1i}) + (F_{1i} V_{kj} - F_{1j} V_{ki}) \end{bmatrix},$$

which by transvecting with g^{ik} and using Lemma 2.1 again along with the fact $g^{ij} V_{ij} = 2|x|^2$ gives

$$\nabla_k \nabla_l U_j^k - \nabla_l \nabla_k U_j^k = -2a \, [nU_{lj} - |x|^2 F_{lj}].$$

Now by substituting (2.17) and the result of the lemma 2.3 in L.H.S. of above equation, we get

$$F_{lj} \nabla_k x^k - F_l^k \nabla_k x_j + \nabla_l \tilde{x}_j - g_{lj} \nabla_k \tilde{x}^k + 2\nabla_k (\rho_l U_j^k) - 2n \nabla_l \tilde{x}_j = 2a \, [|x|^2 F_{li} - nU_{lj}]$$

which, on using (1.2), (2.14), (2.15), (2.16), lemma 2.2 and lemma 2.3 reduces to the form

$$n(\rho_l \tilde{x}_j + \rho_l x_j) = a\,[|x|^2 F_{lj} - (n+1)\, U_{lj}].$$

If $a \neq 0$, then by using the condition $\rho_l \tilde{x}_j + \tilde{\rho}_l x_j = 0$, we get

$$U_{lk} = \frac{|x|^2}{(n+1)} F_{lj} \text{ and hence}$$

$$\nabla_t U_{lj} = 0 \text{ and therefore } \nabla_t V_{lj} = 0, \text{ by virtue of (1.1).} \qquad \ldots (2.19)$$

Operating ∇_t on (2.18) and using (1.1) and (2.19) to get

$$\nabla_t R^h_{kji} = 0.$$

Theorem 2.2 — *In a holomorphically subprojective Kahlerian manifold M_{2n} admitting a local co-ordinate system (x^h), the tensor U_{ij} is recurrent with respect to the recurrent vector field $2(\rho_l + f_{lr} x^r)$ if and only if $\tilde{x}_r + U_{ri}\rho^r = 0$ (or equivalently $x_r + V_{ri}\rho^r = 0$).*

PROOF : Operating ∇_l on (2.1) and using (1.5) and (1.6), we get

$$\nabla_l U_{ij} = \left\{ x_i + (x_i x^r + \tilde{x}_i \tilde{x}^r)\,\rho_r \right\} F_{lj} - \left\{ x_j + (x_j x^r + \tilde{x}_j \tilde{x}^r)\,\rho_r \right\} F_{li}$$
$$+ \left\{ \tilde{x}_j + (\tilde{x}_j x^r - x_j \tilde{x}_r)\,\rho_r \right\} g_{li} - \left\{ \tilde{x}_i + (\tilde{x}_i + x^r - x_i \tilde{x}^r)\,\rho_r \right\} g_{lj}$$
$$+ (\rho_1 + f_{lr} x^r) \left\{ 2(x_i \tilde{x}_j - \tilde{x}_i x_j) \right\},$$

which, by virtue of (2.1) and (2.2) becomes

$$\nabla_l U_{ij} = (x_i + V_{ir}\rho^r)\, F_{lj} - (x_j + V_{jr}\rho^r)\, F_{li} + (\tilde{x}_j + U_{rj}\rho^r)\, g_{li}$$
$$- g_{lj}(\tilde{x}_i + U_{ri}\rho^r) + 2\,(\rho_l + f_{lr} x^r)\, U_{ij}. \qquad \ldots (2.20)$$

Suppose U_{ij} is recurrent with respect to $2\,(\rho_l + f_{lr} x^r)$ as recurrent vector field, then above result gives

$$(x_i + V_{ri}\rho^r)\, F_{lj} - (x_j + V_{rj}\rho^r)\, F_{li} - (\tilde{x}_i + U_{ri}\rho^r)\, g_{lj} + (\tilde{x}_j + U_{rj}\rho^r)\, g_{li} = 0.$$

Transvecting this result by g^{lj} and using (1.2) and the lemma 2.1, we get

$$\tilde{x}_i + U_{ri}\rho^r = 0, \qquad \ldots (2.21)$$

which is equivalent to

$$x_i + V_{ri}\rho^r = 0 \qquad \ldots (2.22)$$

in view of the fact

$$F_l^i(\tilde{x}_i + U_{ri}\rho^r) = F_l^i(F_{ti}x^t + U_{ri}\rho^r) = x_l + V_{rl}\rho^r.$$

Conversely, assume (2.21) so that (2.22) follows. Hence from (2.20) we get

$$\nabla_l U_{ij} = 2(\rho_l + f_{lr}x^r)\, U_{ij}.$$

Note that by substituting (2.1) in the condition (2.21) and using $x_i\tilde{x}^i = 0$, we get $\rho_\Upsilon \tilde{x}^\Upsilon = 0$, which, in view of (2.21) or (2.22), gives $(1 + \rho_r x^r) = 0$. Hence we have the following:

Remark 2.1 : Yamaguchi and Adati[5] have proved the theorem, "The holomorphically subprojective Kahlerian manifold M of the first kind with $(1 + \rho_r x^r) = 0$ and $\rho_r\tilde{x}^r = 0$ is one of the following manifolds :

(i) M is a manifold of constant holomorphic sectional curvature

and (ii) M is locally product manifold of two manifolds of constant holomorphic sectional curvature $H\,(\geq 0)$ and $-H$".

Since the condition $1 + \rho_r x^r = 0$ and $\rho_r\tilde{x}^r = 0$ used in the above theorem are obtained either from (2.21) or (2.22), it follows, from Theorem 2.2, that these conditions can be replaced by the single condition that U_{ij} is recurrent with respect to $2(\rho_l + f_{lr}x^r)$ as recurrent vector field.

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