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STABLE SET OF NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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This paper studies asymptotically or exponentially stable set of nonlinear neutral delay differential equations. Sufficient conditions for stable set are given by the properties of nonnegative matrices and inequality techniques. The main theorems will be illustrated by two examples.

Key Words : Stable Set; Differential Equations; Nonlinear Neutral Delay; Nonnegative Matrix; Inequality

1. INTRODUCTION

In the last few years, the stability theory of differential equations is emerging as an important area of investigation and various results are reported (see [1], [2], [4], [7], [9]). In these results, authors usually studied stability by using Liapunov functional and differential-difference inequalities. However both types of methods are unsatisfactory in a certain sence. Liapunov functional can treat with some nonlinear equations, but it is difficult to construct Liapunov functional (see [2], [4]); On the other hand, equations must be linearized using inequalities and some behaviour of equations might be missed (see [1], [7]). Furthermore, there are multiple equilibriua and globally asymptotic stability results are excluded in nonlinear equations. How far can initial conditions be allowed to vary without disrupting the stability properties established in the immediate vicinity of equilibrium states. On this problem, Hale[2] studied stable set (or manifold) for retarded functional differential equations; Xu *et al.*[8] gave domain of attraction of nonlinear difference systems. But, to authors' knowledge, very few results have been obtained for the stable set of the initial values of nonlinear neutral delay differential equations except that Kolmanovskii and Nosov[4] discussed stability domain for the parameter of linear autonomous neutral functional differential equations.

In this paper, we consider the following nonlinear neutral delay differential equations

$$\dot{x}_i(t) = a_i(x(t), \dot{x}(t))\, x_i(t) + f_i(t, x(t), x(t-\tau(t)), \dot{x}(t-\tau(t))),\ t \ge t_0, \qquad \dots (1)$$

where $i = 1, \dots, n$, $x = [x_1, \dots, x_n]^T \in R^n$, $a_i : R^n \times R^n \to R$ are continuous functions, $\tau(t)$ is nonnegative continuous functions with $0 \le \tau(t) \le \alpha$ ($\alpha \ge 0$ is a constant or $+\infty$) and

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$\lim_{t \to +\infty} (t - \tau(t)) = +\infty, f_i : R \times R^n \times R^n \times R^n \to R$ are continuously differentiable functions and $f_i(t, 0, 0, 0) = 0$.

The initial conditions of eqs. (1) are

$$x_i(t_0 + s) = \phi_i(s), \; -\alpha \le s \le 0, \; i = 1, \ldots, n, \qquad \ldots (2)$$

where $\phi_i(t)$ are continuously differentiable functions. We always assume that (1) and (2) have a solution, denoted by $x(t)$. (see [4] or [6]).

Throughout the paper, $R = (-\infty, +\infty)$, $R_+ = [0, +\infty)$, and $C^1(Y, Z)$ be the class of continuously differential mappings from a topological space Y to a topological space Z. We assume that $C^1 = C^1((-\alpha, 0], R^n)$. For $w \in C^1((-\alpha, 0], R)$, we define

$$\| w \|_\alpha = \sup_{-\alpha \le \theta \le 0} | w(\theta) |, \; \| \dot{w} \|_\alpha = \sup_{-\alpha \le \theta \le 0} | \dot{w}(\theta) |.$$

For $x(t) = [x_1(t), \ldots, x_n(t)]^T \in C^1[R, R^n]$, we employ the notations

$$[x(t), \dot{x}(t)]^+ = [| x_1(t) |, \ldots, | x_n(t) |, | \dot{x}_1(t) |, \ldots, | \dot{x}_n(t) |]^T$$

and

$$[x(t), \dot{x}(t)]_\alpha^+ = [\| x_1(t) \|_\alpha, \ldots, \| x_n(t) \|_\alpha, \| \dot{x}_1(t) \|_\alpha, \ldots, \| \dot{x}_n(t) \|_\alpha]^T,$$

where

$$\| x_i(t) \|_\alpha = \sup_{-\alpha \le \theta \le 0} | x_i(t + \theta) |, \; \| \dot{x}_i(t) \|_\alpha = \sup_{-\alpha \le \theta \le 0} | \dot{x}_i(t + \theta) |.$$

Especially, for any $\phi = [\phi_1, \ldots, \phi_n]^T \in C^1$,

$$[\phi, \dot{\phi}]_\alpha^+ = [\| \phi_1 \|_\alpha, \ldots, \| \phi_n \|_\alpha, \| \dot{\phi}_1 \|_\alpha, \ldots, \| \dot{\phi}_n \|_\alpha]^T.$$

For $r \times r$-matrix $A = [a_{ij}]_{r \times r}$, let

$$\| | A \| | \triangleq \max_{0 \le i \le r} \sum_{j=1}^{r} | a_{ij} |.$$

For vectors A and B, $A \le B$ (or $A < B$) means that each pair of corresponding elements of A and B satisfies this inequality "$\le$" (or "<").

The symbol $\rho(A)$ denotes the spectral radius of a square matrix A.

The following results are given by Horn and Johnson[3] (Theorem 8.3.1, p. 503 and Th. 8.3.2, p 504).

Lemma 1 — If A is a nonnegative square matrix (*i.e.*, $A \ge 0$), then $\rho(A)$ is an eigenvalue of A and there is a nonnegative vector $x \ge 0$ with $x \ne 0$ such that $Ax = \rho(A)x$.

Lemma 2 — Let $A \geq 0, x \in R^r$ and $x \geq 0$ with $x \neq 0$. If $Ax \geq \gamma x$, for some $\gamma \in R$, then $\rho(A) \geq \gamma$.

The notation $W_\rho(A)$ is used to denote the characteristic space associated with $\rho(A)$ (the collection of all $x \in W_\rho(A)$ such that $Ax = \rho(A)x$).

Definition 1 — The set $D \subset C^1$ ($D \backslash \{0\}$ is nonempty) is called to be an asymptotically stable set of (1) if for any $t_0 \geq 0$ and for any $\phi \in D$, there is a continuously nonnegative vector function $v : R^{2n} \to R^{2n}$ with $v(0) = 0$ such that

$$[x(t), \dot{x}(t)]^+ \leq v([\phi, \dot{\phi}]_\alpha^+), \quad t \geq t_0, \qquad \dots (3)$$

and

$$\lim_{t \to +\infty} [x(t), \dot{x}(t)]^+ = 0. \qquad \dots (4)$$

Definition 2 — A set $D \subset C^1$ ($D \backslash \{0\}$ is nonempty) is called to be an exponentially stable set of (1) if for any $t_0 \geq 0$, there exist a constant $\lambda > 0$ and a continuously nonnegative vector function $v : R^{2n} \to R^{2n}$ with $v(0) = 0$ such that for any $\phi \in D$,

$$[x(t), \dot{x}(t)]^+ \leq v([\phi, \dot{\phi}]_\alpha^+)\, e^{-\lambda(t - t_0)}, \quad t \geq t_0. \qquad \dots (5)$$

The purpose of the paper is to study the stable set of nonlinear neutral delay differential equations. In section 2, some sufficient conditions for determining the stable set are given by virtue of the properties of nonnegative parameter matrices and differential inequality. Our theory will be applied to two examples in section 3.

2. Main Results

We make the following assumptions :

(A_1) for $i = 1, 2, \dots, n$ and $t \geq t_0$, there are a positive constant c_i and monotonically nondecreasing continuous functions $a_i^* : R_+^{2n} \to R$ such that $-c_i \leq a_i(x(t), \dot{x}(t)) \leq a_i^*([x(t), \dot{x}(t)]^+) < 0$;

(A_2) for any $i = 1, \dots, n$ and $t \geq t_0$,

$$| f_i(t, x(t), x(t - \tau(t)), \dot{x}(t - \tau(t))) |$$

$$\leq \sum_{j=1}^{n} a_{ij}([x(t), \dot{x}(t)]_\alpha^+) \| x_j(t) \|_\alpha + \sum_{j=1}^{n} b_{ij}([x(t), \dot{x}(t)]_\alpha^+) \| \dot{x}_j(t) \|_\alpha \qquad \dots (6)$$

where $a_{ij}, b_{ij} : R_+^{2n} \to R_+$ are monotonically nondecreasing continuous functions.

In order to describe our main results, we let

$$P(\cdot) = [p_{ij}(\cdot)]_{2n\times 2n} = \begin{bmatrix} \frac{\bar{a}_{11}(\cdot)}{c_1} & \cdots & \frac{\bar{a}_{1n}(\cdot)}{c_1} & \frac{b_{11}(\cdot)}{c_1} & \cdots & \frac{b_{1n}(\cdot)}{c_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\bar{a}_{n1}(\cdot)}{c_n}1 & \cdots & \frac{\bar{a}_{nn}(\cdot)}{c_n} & \frac{b_{n1}(\cdot)}{c_n} & \cdots & \frac{b_{nn}(\cdot)}{c_n} \\ \bar{\bar{a}}_{11}(\cdot) & \cdots & \bar{\bar{a}}_{1n}(\cdot) & b_{11}(\cdot) & \cdots & b_{1n}(\cdot) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{\bar{a}}_{n1}(\cdot) & \cdots & \bar{\bar{a}}_{nn}(\cdot) & b_{n1}(\cdot) & \cdots & b_{nn}(\cdot) \end{bmatrix}$$

$$\bar{a}_{ij}(\cdot) = \begin{cases} c_i + a_i^*(\cdot) + a_{ii}(\cdot), & \text{if } i = j, \\ a_{ij}(\cdot), & \text{if } i \neq j, \end{cases} \quad \bar{\bar{a}}_{ij}(\cdot) = \begin{cases} c_i + a_{ii}(\cdot), & \text{if } i = j, \\ a_{ij}(\cdot), & \text{if } i \neq j. \end{cases}$$

Theorem 1 — *In addition to the assumptions (A_1) and (A_2), if the set $D_1\backslash\{0\}$* is nonempty, where

$$D_1 = \{\phi \in C^1 : [\phi, \dot{\phi}]_\alpha^+ < K \in R_+^{2n}, [\phi, \dot{\phi}]_\alpha^+ \in W_\rho(P(K)), \rho(P(K)) < 1\}. \quad \text{... (7)}$$

Then D_1 is an asymptotically stable set of (1).

PROOF : From (1), we have

$$\frac{d}{dt}|x_i(t)| = \frac{d}{dt}x_i(t) \operatorname{sign} x_i(t)$$

$$= a_i(x(t), \dot{x}(t))\, x_i(t) \operatorname{sign} x_i(t) + f_i(t, x(t), x(t-\tau(t)), \dot{x}(t-\tau(t))) \operatorname{sign} x_i(t)$$

$$\leq -c_i|x_i(t)| + [a_i(x(t), \dot{x}(t)) + c_i]\,|x_i(t)| + |f_i(t, x(t), x(t-\tau(t)), \dot{x}(t-\tau(t)))|. \quad \text{... (8)}$$

So, from assumptions (A_1), (A_2) and (8),

$$|x_i(t)| \leq e^{-c_i(t-t_0)}|\phi_i(0)| + \int_{t_0}^{t} \{[a_i(x(s), \dot{x}(s)) + c_i]|x_i(s)|$$

$$+ |f_i(s, x(s), x(s-\tau(s)), \dot{x}(s-\tau(s)))|\}\, e^{-c_i(t-s)}\, ds$$

$$\leq e^{-c_i(t-t_0)}|\phi_i(0)| + \int_{t_0}^{t} \{[a_i^*([x(s), \dot{x}(s)]^+) + c_i]/x_i(s)|$$

$$+ \sum_{j=1}^{n} a_{ij}([x(s), \dot{x}(s)]_{\alpha}^{+} \| x_j(s) \|_{\alpha}$$

$$+ \sum_{j=1}^{n} b_{ij}([x(s), \dot{x}(s)]_{\alpha}^{+}) \| \dot{x}_j(s) \|_{\alpha}\} \quad e^{c_i(t-s)} ds. \qquad \dots (9)$$

By (1) and assumptions (A_1), (A_2), we obtain

$$|\dot{x}_i(t)| \le |a_i(x(t), \dot{x}(t)) \| x_i(t)| + |f_i(t, x(t), x(t-\tau(t)), \dot{x}(t-\tau(t)))|$$

$$\le c_i |x_i(t)| + \sum_{j=1}^{n} a_{ij}([x(t), \dot{x}(t)]_{\alpha}^{+}) \| x_j(t) \|_{\alpha} + \sum_{j=1}^{n} b_{ij}([x(t), \dot{x}(t)]_{\alpha}^{+}) \| \dot{x}_j(t) \|_{\alpha} \quad \dots (10)$$

If there is a positive vector such that $\rho(P(K)) < 1$, applying [5, Theorem 9.16], we can find positive numbers δ_i $(i = 1, \dots, 2n)$ such that

$$\sum_{j=1}^{2n} \delta_i \delta_j^{-1} p_{ij}(K) < 1. \qquad \dots (11)$$

When $\phi \in D_1$ (that is, there is a positive vector K such that $[\phi, \dot{\phi}]_{\alpha}^{+} < K$, $[\phi, \dot{\phi}]_{\alpha}^{+} \in W_{\rho}(P(K))$ and $\rho(P(K)) < 1$), there exists a sufficient small positive constant ε such that

$$[\phi, \dot{\phi}]_{\alpha}^{+} + \beta\varepsilon < K,$$

where $\beta = [\delta_1^{-1}, \dots, \delta_{2n}^{-1}]^T$ and δ_i are defined as (11).

We first prove that

$$[x(t), \dot{x}(t)]^{+} < [\phi, \dot{\phi}]_{\alpha}^{+} + \beta\varepsilon < K, \quad t \ge t_0. \qquad \dots (12)$$

If (12) is not true, then there must be some $l \in \{1, \dots, n\}$ and $t_1 > t_0$ such that

$$\text{(I)} \quad |x_l(t_1)| = \| \phi_l \|_{\alpha} + \delta_l^{-1} \varepsilon, \ |x_l(t)| < \| \phi_l \|_{\alpha} + \delta_l^{-1} \varepsilon \text{ for } t < t_1,$$

$$[x(t), \dot{x}(t)]^{+} \le [\phi, \dot{\phi}]_{\alpha}^{+} + \beta\varepsilon < K \text{ for } t \le t_1, \qquad \dots (13)$$

or

$$\text{(II)} \quad |\dot{x}_l(t_1)| = \| \dot{\phi}_l \|_{\alpha} + \delta_{n+l}^{-1} \varepsilon, \ |\dot{x}_l(t)| < \| \dot{\phi}_l \|_{\alpha} + \delta_{n+l}^{-1} \varepsilon \text{ for } t < t_1,$$

$$[x(t), \dot{x}(t)]^{+} \le [\phi, \dot{\phi}]_{\alpha}^{+} + \beta\varepsilon < K \text{ for } t \le t_1. \qquad \dots (14)$$

From $[\phi, \dot{\phi}]_\alpha^+ \in W_\rho(P(K))$, we have $P(K)[\phi, \dot{\phi}]_\alpha^+ = \rho(P(K))[\phi, \dot{\phi}]_\alpha^+$, that is

$$\left[\sum_{j=1}^{n} \bar{a}_{lj}(K) \| \phi_j \|_\alpha + \sum_{j=1}^{n} b_{lj}(K) \| \dot{\phi}_j \|_\alpha \right] a_l^{-1} = \rho(P(K)) \| \phi_l \|_\alpha, \qquad \text{... (15)}$$

and

$$\sum_{j=1}^{n} \bar{\bar{a}}_{lj}(K) \| \phi_j \|_\alpha + \sum_{j=1}^{n} b_{lj}(K) \| \dot{\phi}_j \|_\alpha = \rho(P(K)) \| \dot{\phi}_l \|_\alpha. \qquad \text{... (16)}$$

For case (I), from (9), (11), (13), (15), assumptions (A_1), (A_2) and $\rho(P(K)) < 1$,

$$\begin{aligned}
\| \phi_l \|_\alpha + \delta_l^{-1} \varepsilon &= | x_l(t_1) | \\
&\le e^{-c_l(t_1 - t_0)} | \phi_l(0) | + \int_{t_0}^{t_1} \Bigg[[a_l^*([x(s), \dot{x}*(s)]^+) + c_l] | x_l(s) | \\
&\quad + \sum_{j=1}^{n} a_{lj}([x(s), \dot{x}(s)]_\alpha^+) \| x_j(s) \|_\alpha + \sum_{j=1}^{n} b_{lj}[(x(s), \dot{x}(s)]_\alpha^+) \| \dot{x}_j(s) \|_\alpha \Bigg] e^{-c_l(t-s)} ds \\
&\le e^{-c_l(t_1 - t_0)} | \phi_l(0) | + \int_{t_0}^{t_1} \Bigg[\sum_{j=1}^{n} \bar{a}_{lj}(K)(\| \phi_j \|_\alpha + \delta_j^{-1} \varepsilon) \\
&\qquad + \sum_{j=1}^{n} b_{lj}(K)(\| \dot{\phi}_j \|_\alpha + \delta_{n+j}^{-1} \varepsilon) \Bigg] e^{-c_l(t-s)} ds \\
&\le e^{-c_l(t_1 - t_0)} \| \phi_l \|_\alpha + \Bigg[\sum_{j=1}^{n} \bar{a}_{lj}(K)(\| \phi_j \|_\alpha + \delta_j^{-1} \varepsilon) \\
&\qquad + \sum_{j=1}^{n} b_{lj}(K)(\| \dot{\phi}_j \|_\alpha + \delta_{n+j}^{-1} \varepsilon) \Bigg] \frac{1 - e^{-c_l(t_1 - t_0)}}{c_l} \\
&= e^{-c_l(t_1 - t_0)} \| \phi_l \|_\alpha + (1 - e^{-c_l(t_1 - t_0)}) \rho(P(K)) \| \phi_l \|_\alpha \\
&\qquad + \frac{1 - e^{-c_l(t_1 - t_0)}}{c_l} \left[\sum_{j=1}^{n} \bar{a}_{lj}(K) \delta_j^{-1} + \sum_{j=1}^{n} b_{lj}(K) \delta_{n+j}^{-1} \right] \varepsilon \\
&< \| \phi_l \|_\alpha + \delta_l^{-1} \varepsilon,
\end{aligned}$$

which is a contradiction.

For case (II), from (10), (11), (14), (16), assumptions (A_1), (A_2) and $\rho(P(K)) < 1$,

$$\delta_{n+l}^{-1}\varepsilon + \| \dot{\phi}_l \|_\alpha = |\dot{x}_l(t_1)|$$

$$\leq c_l |x_l(t_1)| + \sum_{j=1}^{n} a_{lj}([x(t_1), \dot{x}(t_1)]_\alpha^+) \| x_j(t_1) \|_\alpha + \sum_{j=1}^{n} b_{lj}([x(t_1), \dot{x}(t_1)]_\alpha^+) \| \dot{x}_j(t_1) \|_\alpha$$

$$\leq \sum_{j=1}^{n} \bar{\bar{a}}_{lj}(K)(\| \phi_j \|_\alpha + \delta_j^{-1}\varepsilon) + \sum_{j=1}^{n} b_{lj}(K)(\| \dot{\phi}_j \|_\alpha + \delta_{n+j}^{-1}\varepsilon)$$

$$\leq \rho(P(K)) \| \dot{\phi}_l \|_\alpha + \sum_{j=1}^{n} \bar{\bar{a}}_{lj}(K)\delta_j^{-1}\varepsilon + \sum_{j=1}^{n} b_{lj}(K)\delta_{n+j}^{-1}\varepsilon$$

$$< \| \dot{\phi}_l \|_\alpha + \delta_{n+l}^{-1}\varepsilon,$$

which is also a contradiction. So (12) holds. Letting $\varepsilon \to 0$, we get

$$[x(t), \dot{x}(t)]^+ \leq [\phi, \dot{\phi}]_\alpha^+ < K, \; t \geq t_0. \qquad \dots (17)$$

We now prove that

$$\lim_{t \to +\infty} [x(t), \dot{x}(t)]^+ = 0. \qquad \dots (18)$$

From (17), there is a nonnegative constant vector $\sigma = [\sigma_1, \dots, \sigma_{2n}]^T$ such that

$$\lim_{t \to +\infty} \sup |x_i(t)| = \sigma_i \leq K_i \text{ and } \lim_{t \to +\infty} \sup |\dot{x}_i(t)| = \sigma_{n+i} \leq K_{n+i}, \qquad \dots (19)$$

where $i = 1, \dots, n$ and K_j is jth component of vector $\mathbf{K}$.

According to definition of limsup and $\lim_{t \to +\infty} (t - \tau(t)) = +\infty$, for sufficient small constant $\varepsilon > 0$, there is $t_2 > t_0$ such that, for any $t \geq t_2$,

$$[x_i(t - \tau(t)), \dot{x}(t - \tau(t))]^+ \leq (1 + \varepsilon)\sigma. \qquad \dots (20)$$

For the above ε, taking $T > \max_{1 \leq i \leq n} \left\{ -\frac{\ln c_i \varepsilon}{c_i} \right\}$, we have

$$\int_T^{+\infty} e^{-c_i s} ds < \varepsilon, \; i = 1, \dots, n. \qquad \dots (21)$$

So, by (9), (17), (20) and (21), when $t > t_2 + T$,

$$|x_i(t)| \le e^{-c_i(t-t_0)} \| \phi_i(0) \| + \left(\int_{t_0}^{t-T} + \int_{t-T}^{t} \right) e^{-c_i(t-s)} \Big\{ [c_i + a_i^*([x(t), \dot{x}(t)]^+)] \, |x_i(s)|$$

$$+ \left\{ \sum_{j=1}^{n} a_{ij}([x(s), \dot{x}(s)]_\alpha^+) \| x_j(s) \|_\alpha + \sum_{j=1}^{n} b_{ij}([x(s), \dot{x}(s)]_\alpha^+) \| \dot{x}_j(s) \|_\alpha \right\}$$

$$\le e^{-c_i(t-t_0)} |\phi_i(0)| + \int_{-\infty}^{t-T} e^{-c_i(t-s)} \left\{ \sum_{j=1}^{n} \bar{a}_{ij}(K) K_j + \sum_{j=1}^{n} b_{ij}(K) K_{n+j} \right\} ds$$

$$+ \int_{t-T}^{t} e^{-c_i(t-s)} \left\{ \sum_{j=1}^{n} \bar{a}_{ij}(K)(1+\varepsilon)\sigma_j + \sum_{j=1}^{n} b_{ij}(K)(1+\varepsilon)\sigma_{n+j} \right\} ds$$

$$< e^{-c_i(t-t_0)} |\phi_i(0)| + \left\{ \sum_{j=1}^{n} \bar{a}_{ij}(K) K_j + \sum_{j=1}^{n} b_{ij}(K) K_{n+j} \right\} \varepsilon$$

$$+ \frac{1}{c_i} \left\{ \sum_{j=1}^{n} \bar{a}_{ij}(K)(1+\varepsilon)\sigma_j + \sum_{j=1}^{n} b_{ij}(K)(1+\varepsilon)\sigma_{n+j} \right\}$$

Thus, letting $t \to +\infty$ and $\varepsilon \to 0$, then

$$\sigma_i \le \frac{1}{c_i} \left[\sum_{j=1}^{n} \bar{a}_{ij}(K)\sigma_j + \sum_{j=1}^{n} b_{ij}(K)\sigma_{n+j} \right]. \qquad \text{... (22)}$$

Similarly, by (10), (17) and (20), we can obtain that, for $t \ge t_2$,

$$|\dot{x}_i(t)| \le \sum_{j=1}^{n} \bar{\bar{a}}_{ij}(K)(1+\varepsilon)\sigma_j + \sum_{j=1}^{n} b_{ij}(K)(1+\varepsilon)\sigma_{n+j}.$$

So, letting $t \to +\infty$ and $\varepsilon \to 0$, then

$$\sigma_{n+i} \le \sum_{j=1}^{n} \bar{\bar{a}}_{ij}(K)\sigma_j + \sum_{j=1}^{n} b_{ij}(K)\sigma_{n+j}. \qquad \text{... (23)}$$

Combining with (22) and (23), if $\sigma \ge 0$ and $\sigma \ne 0$, then by Lemma 2,

$$\rho(P(K)) \ge 1.$$

This contradicts $\rho(P(K)) < 1$. Hence, (18) holds and the proof of this Theorem is complete.

Theorem 2 — *Assume that* (A_1) *and* (A_2) *hold and the set* $D_2 \setminus \{0\}$ *is nonempty, where*

$$D_2 = \{\phi \in C^1 : [\phi, \dot{\phi}]_\alpha^+ < k\boldsymbol{E}_{2n}, ||| P(k\boldsymbol{E}_{2n}) ||| < 1\} \quad \text{... (24)}$$

and $\boldsymbol{E}_{2n} = [1, ..., 1]^T$ *is a 2n-dimensional vector.*

Then D_2 *is an asymptotically stable set of (1).*

PROOF : First, we show that when $\phi \in D_2$ (that is, there is a positive constant k such that $[\phi, \phi]_\alpha^+ < kE_{2n}$ and $||| P(kE_{2n}) ||| < 1$),

$$[\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)]^+ < k\boldsymbol{E}_{2n}, \; t \geq t_0. \quad \text{... (25)}$$

If (25) is not true, then there must be some $l \in \{1, ..., n\}$ and $t_1 > t_0$ such that

$$\text{(III)} \;\; |\boldsymbol{x}_l(t_1)| = k, \; |\boldsymbol{x}_l(t)| < k \text{ for } t < t_1, \; [\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)]^+ \leq k\boldsymbol{E}_{2n} \text{ for } t \leq t_1, \quad \text{... (26)}$$

or

$$\text{(IV)} \;\; |\dot{\boldsymbol{x}}_l(t_1)| = k, \; |\dot{\boldsymbol{x}}_l(t)| < k \text{ for } t < t_1, \; [\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)]^+ \leq k\boldsymbol{E}_{2n} \text{ for } t \leq t_1. \quad \text{... (27)}$$

For case (III), from (9), (26), assumptions (A_1), (A_2) and $||| P(k\boldsymbol{E}_{2n}) ||| < 1$, we have that

$$\begin{aligned} k &= |\boldsymbol{x}_l(t_1)| \\ &\leq e^{-c_l(t_1 - t_0)} |\phi_l(0)| + \int_{t_0}^{t_1} \left[\sum_{j=1}^{n} \bar{\boldsymbol{a}}_{lj}(k\boldsymbol{E}_{2n}) + \sum_{j=1}^{n} \boldsymbol{b}_{lj}(k\boldsymbol{E}_{2n}) \right] k e^{-c_l(t_1 - s)} ds \\ &= e^{-c_l(t_1 - t_0)} k + \frac{1 - e^{-c_l(t_1 - t_0)}}{c_l} \left[\sum_{j=1}^{n} \bar{\boldsymbol{a}}_{lj}(k\boldsymbol{E}_{2n}) + \sum_{j=1}^{n} \boldsymbol{b}_{lj}(k\boldsymbol{E}_{2n}) \right] k \\ &< e^{-c_l(t_1 - t_0)} k + (1 - e^{-c_l(t_1 - t_0)}) k = k, \end{aligned}$$

which is a contradiction.

For case (IV), from (1), (27), assumptions (A_1), (A_2) and $||| P(kE_{2n}) ||| < 1$,

$$\begin{aligned} k &= |\dot{\boldsymbol{x}}_l(t_1)| \\ &\leq \sum_{j=1}^{n} \bar{\bar{\boldsymbol{a}}}_{lj}(k\boldsymbol{E}_{2n}) k + \sum_{j=1}^{n} \boldsymbol{b}_{lj}(k\boldsymbol{E}_{2n}) k \\ &< k, \end{aligned}$$

which is also a contradiction. So (25) holds.

Secondly, we prove that

$$\lim_{t \to +\infty} [x(t), \dot{x}(t)]^+ = 0. \qquad \text{... (28)}$$

From (25), there is a nonnegative constant vector $\sigma = [\sigma_1, \ldots, \sigma_{2n}]^T$ such that

$$\lim_{t \to +\infty} \sup |x_i(t)| = \sigma_i \le k \text{ and } \lim_{t \to +\infty} \sup |\dot{x}_i(t)| = \sigma_{n+i} \le k, \qquad \text{... (29)}$$

where $i = 1, \ldots, n$.

So, by (9), (20), (21) and (25), when $t > t_2 + T$,

$$|x_i(t)| \le e^{-c_i(t-t_0)} |\phi_i(0)| + \left(\int_{t_0}^{t-T} + \int_{t-T}^{t} \right) \Bigg[[c_i + a_i^*([x(t), \dot{x}(t)]^+)] |x_i(s)|$$

$$+ \sum_{j=1}^{n} a_{ij}([x(s), \dot{x}(s)]_\alpha^+) \| x_j(s) \|_\alpha + \sum_{j=1}^{n} b_{ij}([x(s), \dot{x}(s)]_\alpha^+) \| \dot{x}_j(s) \|_\alpha \Bigg] e^{-c_i(t-s)} ds$$

$$\le e^{-c_i(t-t_0)} |\phi_i(0)| + \int_{-\infty}^{t-T} \left\{ \sum_{j=1}^{n} \bar{a}_{ij}(kE_{2n}) k + \sum_{j=1}^{n} b_{ij}(kE_{2n}) k \right\} e^{-c_i(t-s)} ds$$

$$+ \int_{t-T}^{t} \left\{ \sum_{j=1}^{n} \bar{a}_{ij}(kE_{2n})(1+\varepsilon)\sigma_j + \sum_{j=1}^{n} b_{ij}(kE_{2n})(1+\varepsilon)\sigma_{n+j} \right\} e^{-c_i(t-s)} ds$$

$$< e^{-c_i(t-t_0)} |\phi_i(0)| + \left\{ \sum_{j=1}^{n} \bar{a}_{ij}(kE_{2n}) k + \sum_{j=1}^{n} b_{ij}(kE_{2n}) k \right\} \varepsilon$$

$$+ \frac{1}{c_i} \left\{ \sum_{j=1}^{n} \bar{a}_{ij}(kE_{2n})(1+\varepsilon)\sigma_j + \sum_{j=1}^{n} b_{ij}(kE_{2n})(1+\varepsilon)\sigma_{n+j} \right\}.$$

Thus, letting $t \to +\infty$ and $\varepsilon \to 0$, then

$$\sigma_i \le \frac{1}{c_i} \left\{ \sum_{j=1}^{n} \bar{a}_{ij}(kE_{2n})\sigma_j + \sum_{j=1}^{n} b_{ij}(kE_{2n})\sigma_{n+j} \right\}. \qquad \text{... (30)}$$

So,

$$\max_{1 \le i \le n} \sigma_i \le \max_{1 \le i \le n} \left\{ \frac{1}{c_i} \left\{ \sum_{j=1}^{n} \bar{a}_{ij}(kE_{2n})\sigma_j + \sum_{j=1}^{n} b_{ij}(kE_{2n})\sigma_{n+j} \right\} \right\}. \qquad \text{... (31)}$$

Similarly, by (10), (20) and (25), we can obtain that, for $t \geq t_2 + T$,

$$|\dot{x}_i(t)| \leq \sum_{j=1}^{n} \bar{\bar{a}}_{ij}(kE_{2n})(1+\varepsilon)\sigma_j + \sum_{j=1}^{n} b_{ij}(kE_{2n})(1+\varepsilon)\sigma_{n+j}.$$

So, letting $t \to +\infty$ and $\varepsilon \to 0$, then

$$\sigma_{n+i} \leq \sum_{j=1}^{n} \bar{\bar{a}}_{ij}(kE_{2n})\sigma_j + \sum_{j=1}^{n} b_{ij}(kE_{2n})\sigma_{n+j} \quad \dots (32)$$

and

$$\max_{1 \leq i \leq n} \sigma_{n+i} \leq \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} \bar{\bar{a}}_{ij}(kE_{2n})\sigma_j + \sum_{j=1}^{n} b_{ij}(kE_{2n})\sigma_{n+j} \right\}. \quad \dots (33)$$

Letting $\sigma_M = \max_{1 \leq i < 2n} \sigma_i$, by (31) and (33),

$$\sigma_M \leq \max_{1 \leq i \leq n} \frac{1}{c_i} \left\{ \sum_{j=1}^{n} \bar{a}_{ij}(kE_{2n}) + \sum_{j=1}^{n} b_{ij}(kE_{2n}) \right\} \sigma_M$$

and

$$\sigma_M \leq \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} \bar{\bar{a}}_{ij}(kE_{2n}) + \sum_{j=1}^{n} b_{ij}(kE_{2n}) \right\} \sigma_M.$$

Noticing that $|||P(kE_{2n})||| < 1$, it is obtained that $\sigma_M = 0$. Hence (28) holds and the proof is completed.

Let $y_i = d_i^{-1}|x_i|$ $(i = 1, \dots, n)$ and $y_i = d_i^{-1}|\dot{x}_{i-n}|$ $(i = n+1, \dots, 2n)$, or $[x, \dot{x}]^+ = \text{diag}\{d_1, \dots, d_{2n}\}[y_1, \dots, y_{2n}]^T \triangleq Dy$, where $d_i > 0, D = \text{diag}\{d_1, \dots, d_{2n}\}$ is diagonal matrix with i-diagonal element d_i and $y = [y_1, \dots, y_{2n}]^T$.

Then, from (9) and (10), it is obtained that

$$y_i(t) \leq e^{-c_i(t-t_0)}|\phi_i(t_0)|d_i^{-1} + d_i^{-1}\int_{t_0}^{t}\Bigg[[a_i^*(Dy(s)) + c_i]d_i y_i(s)$$

$$+ \sum_{j=1}^{n} a_{ij}(D[y(s)]_\alpha^+)d_j\|y_j(s)\|_\alpha + \sum_{j=1}^{n} b_{ij}(D[y(s)]_\alpha^+)d_{j+n}\|y_{n+j}(s)\|_\alpha\Bigg]e^{-c_i(t-s)}ds \quad \dots (34)$$

and $$y_{n+i}(t) \le c_i y_i(t) + \sum_{j=1}^{n} d_i^{-1} a_{ij}(D[y(t)]_\alpha^+) d_j \| y_j(t) \|_\alpha$$

$$+ \sum_{j=1}^{n} d_i^{-1} b_{ij}(D[y(t)]_\alpha^+) d_{n+j} \| y_{n+j}(t) \|_\alpha. \quad \ldots (35)$$

Similar to proof of Theorem 2, we can obtain the follow corollaries.

Corollary 1 — Assume that (A_1) and (A_2) hold and there are a positive constant vector $K \in R^{2n}$ and positive constants d_i $(1 = 1, \ldots, 2n)$ such that

$$\sum_{j=1}^{2n} d_i^{-1} d_j p_{ij}(K) < 1.$$

If the set $D_2' \backslash \{0\}$ is nonempty, where

$$D_2' = \{\phi \in C^1 : [\phi, \dot{\phi}]_\alpha^+ < \bar{K} \text{ diag } \{d_1, \ldots, d_{2n}\}\}, \quad \ldots (36)$$

$\bar{K} = \min \{d_i^{-1} K_i\}$ and K_i is ith component of K.

Then D_2' is an asymptotically stable set of (1).

Corollary 2 — In addition to (A_1) and (A_2), if there is a positive constant vector **K** such that

$$\rho(P(K)) < 1.$$

Then zero of (1) is asymptotically stable.

Theorem 3 — *Under the assumptions of Theorem 1 and Theorem 2, if the boundedness α of delay is a bounded constant, then $D_1 \cup D_2$ is an exponentially stable set of (1), where D_1 and D_2 are given as Theorem 1 and Theorem 2.*

PROOF : From the property that the eigenvalues of a square matrix depend continuously upon its entries [3, p. 540], for all $i = 1, 2, \ldots, n$, there must be a sufficient small positive number $\lambda < \min \{c_1, \ldots, c_n\}$ such that

$$\rho(P(K)\, e^{\lambda\alpha}) \frac{c_i}{c_i - \lambda} < 1. \quad \ldots (37)$$

From (9), we have that

$$| x_i(t) | \, e^{\lambda(t - t_0)} \le e^{-(c_i - \lambda)(t - t_0)} | \phi_i(0) | + e^{\lambda\alpha} \int_{t_0}^{t}$$

$$\Big\{ | a_i^* ([x(s), \dot{x}(s)]^+) + c_i] \, \| x_i(s) e^{\lambda(s-t_0)} \|_\alpha$$

$$+ \sum_{j=1}^{n} a_{ij} ([x(s), \dot{x}(s)]_\alpha^+) \, \| x_j(s) e^{\lambda(s-t_0)} \|_\alpha$$

$$+ \left\{ \sum_{j=1}^{n} b_{ij} ([x(s), \dot{x}(s)]_\alpha^+) \, \| \dot{x}_j(s) e^{\lambda(s-t_0)} \|_\alpha \right\} e^{-(c_i - \lambda)(t-s)} \, ds.$$

From (10), we have that

$$| \dot{x}_i(t) | e^{\lambda(t-t_0)} \le c_i | x_i(t) | e^{\lambda(t-t_0)} + e^{\lambda\alpha} \sum_{j=1}^{n} a_{ij} ([x(t), \dot{x}(t)]_\alpha^+) \, \| x_j(t) e^{\lambda(t-t_0)} \|_\alpha$$

$$+ e^{\lambda\alpha} \sum_{j=1}^{n} b_{ij} ([x(t), \dot{x}(t)]_\alpha^+) \, \| \dot{x}_j(t) e^{\lambda(t-t_0)} \|_\alpha.$$

The remainder of the proof is essentially the same as the proof of Theorem 1 and Theorem 2 and omitted.

3. Applications

As an application, we shall discuss in this section, two examples to demonstrate the advantages of our results.

Example 1 — Consider neutral delay differential equation

$$\dot{x}(t) = -2x(t) + x(t)\, x(t - \tau(t))\, \dot{x}(t - \tau(t)), \qquad \text{... (38)}$$

where

$$\lim_{t \to +\infty} (t - \tau(t)) = +\infty.$$

Here $n = 1$, $a_{11} = -2$ and $b_{11}(K) = K_1^2$ with $K = [K_1, K_2]^T$. If we take that $c_1 = 2, a_{11}^* = -2$, then

$$P(K) = \begin{bmatrix} 0 & \dfrac{K_1^2}{2} \\ 2 & K_1^2 \end{bmatrix}.$$

By Theorem 1, we obtain

$$D_1 = \{ \phi \in C^1 : [\phi, \dot{\phi}]_\alpha^+ \in \Omega^+ \},$$

where
$$\Omega^{+} = \left\{ [y_1, y_2]^T : 0 < y_1 \left\langle \frac{\sqrt{2}}{2}, y_2 \right\rangle 4y_1, y_2 < \left(2 + \frac{4}{y_1^2 + \sqrt{y_1^4 + 4y_1^2}} \right) y_1 \right\}.$$

If we take that $d_1 = 1, d_2 = 4, K_1 < \frac{\sqrt{2}}{2}$, we obtained

$$d_1^{-1}\left(d_1 \cdot 0 + d_2 \cdot \frac{K_1^2}{2} \right) < 1 \text{ and } d_2^{-1}(2d_1 + d_2 k_1^2) < 1.$$

By Corollary 1,

$$D_2' = \left\{ \phi \in C^1 : [\phi, \dot{\phi}]_{\alpha}^{+} \leq \frac{\sqrt{2}}{2} [1, 4]^T \right\}.$$

Thus $D_1 \cup D_2'$ is an asymptotically stable set of (38).

Example 2 — Consider the equations

$$\left\{ \begin{aligned} \dot{x}_1(t) &= -\frac{x_1(t)}{2 + x_1^2(t)} + x_1(t)\, x_1^2(t-\tau) + x_1(t-\tau)\, x_2(t)\, \dot{x}_2(t-\tau) \\ \dot{x}_2(t) &= x_1(t)\, x_2^2(t) - \frac{1}{4} x_2(t) + \frac{1}{2} x_2(t)\, x_2(t-\tau) + \dot{x}_1^3(t-\tau), \end{aligned} \right\} \quad \text{... (39)}$$

where τ is a constant.

Taking $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{4}$, we have

$$P(kE_4) = \begin{bmatrix} 1 - \frac{2}{1+k^2} + 2k^2 & 0 & 0 & 2k^2 \\ 4k^2 & 2k & 4k^2 & 0 \\ \frac{1}{2} + k^2 & 0 & 0 & k^2 \\ k^2 & \frac{1}{4} + \frac{1}{2}k & k^2 & 0 \end{bmatrix}. \quad \text{... (40)}$$

From $|||\, P(kE_4)\, ||| < 1$ and Theorem 3, it is obtained that

$$D_2 = \{ \phi \in C^1 : [\phi, \dot{\phi}(t)]_{\alpha}^{+} < 0.25\, E_4 \}$$

is an exponentially stable set.

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