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SOME RESULTS ABOUT INTERPOLATION WITH NODES ON THE UNIT CIRCLE

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In this paper, convergence of interpolating Laurent polynomials to a function defined on the unit circle is studied. For this purpose, zeros of certain polynomials associated with Szegö polynomials are used as interpolation nodes.

Key Words : Interpolation Nodes; Laurent Polynomials; Szegö Polynomials; Continuous Functions; Unit Circle; Chelyshev System; Endös-Turan Theorem

1. INTRODUCTION

For a given function defined on a finite interval of the real line, the problem of studying convergence of sequences of interpolating polynomials has been extensively considered throughout this last century. In this paper, we shall be concerned with functions defined on the unit circle, which will be denoted, on the sequel, as $\mathbb{T} = \{z : |z| = 1\}$. Since a continuous function can be uniformly approximated by Laurent polynomials, (see Corollary 2.2) i.e. functions of the form: $L(z) = \sum_{j=-p}^{q} \alpha_j z^j$, p and q being nonnegative integers, it seems natural to study the convergence of sequences of interpolating Laurent polynomials. On the sequel, $\Lambda_{-p,q}$ will denote the space of Laurent polynomials, as given above. On the other hand, Π_k $(k \in \mathbb{Z}; k \geq 0)$, represents the space of polynomials of degree k at most, Π the space of all polynomials and Λ the space of all Laurent polynomials (L-polynomials). (Observe that $\Pi_k = \Lambda_{0,k}$).

Apart this theoretical motivation, our main interest in considering this problem arises when one wishes to estimate an integral of the type $I_\omega = \int_0^{2\pi} f(e^{i\theta})\,\omega(\theta)\,d\theta$, ($\omega(\theta)$ being an L_1-integrable function) by means of a quadrature formula of the form $I_n(f) = \sum_{j=1}^{n} A_j f(x_j)$ with $x_i \neq x_j$ and

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$x_j \in \mathbb{T}$, $j = 1, \ldots, n$. The most usual criterium to determine the coefficients or weights $\{A_j\}_{j=1}^{n}$ along with nodes $\{x_j\}_{j=1}^{n}$ is imposing that $I_n(f)$ exactly integrates as many L-polynomials as possible. On the other hand, since $\{x^j\}_{j=-p}^{q}$, $(p+q=n-1)$ is a Chebyshev system on any set $X \in \mathbb{C}$ such that $0 \notin X$, then, given n distinct nodes $\{x_j\}_{j=1}^{n}$ on $\mathbb{T}$, there exists a unique L-polynomial $R_n(f, x)$ in $\Lambda_{-p,q}$ such that

$$R_n(f, x_j) = f(x_j), j = 1, \ldots, n.$$

Furthermore, one can write $R_n(f, x) = \sum_{j=1}^{n} L_j(x) f(x_j)$ where $L_j \in \Lambda_{-p,q}$ and $L_j(x_k) = \delta_{j,k}$. Hence,

$$L_\omega(R_n(f, \cdot)) = \sum_{j=1}^{n} I_\omega(L_j) f(x_j) = \sum_{j=1}^{n} A_j f(x_j) = I_n(f)$$

and obviously $I_n(f) = I_\omega(f)$ for any $f \in \Lambda_{-p,q}$.

Thus, we see that, the error in the quadrature formula is essentially dominated by the error in the interpolation, i.e.

$$|I_\omega(f) - I_n(f)| \le \int_0^{2\pi} |f(x) - R_n(f, x)|\,|\omega(x)|\,dx. \qquad \ldots (1.1)$$

By (1.1) we also see that, from the L_p-convergence $(p > 1)$ for $R_n(f, x)$, convergence of the sequence of quadrature formulas can be deduced. As for choices of interpolation nodes, it is well known that most of the contributions make use of the roots of the unity. In this paper, zeros of the so-called para-orthogonal polynomials will be used, both in Section 2 dealing with continuous functions on $\mathbb{T}$ and in Section 3, where analytic functions on a region containing $\mathbb{T}$ are considered.

2. Interpolation of Continuous Functions

From the Erdös-Turan Theorem (see [1]) we know that if f is a real continuous function with period 2π, i.e. $f \in C_{2\pi}$, and if $t_n(\theta)$ denotes the unique trigonometric polynomial of order n that coincides with $f(\theta)$ in $(2n + 1)$ points equally spaced over an interval of length 2π, then $t_n(\theta)$ convergence to $f(\theta)$ on that interval in the L_2 - *norm* with respect to the Lebesgue measure, i.e.

$$\lim_{n \to \infty} \int_0^{2\pi} |f(\theta) - t_n(\theta)|^2\, d\theta = 0. \qquad \ldots (2.1)$$

On the other hand, Walsh and Sharma[2] proved an analogous result in the complex domain. More precisely, let the function $f(z)$ be analytic in $\mathbb{D}$, continuous in $\mathbb{D} \cup \mathbb{T}$ and let p_n be the polynomial of degree n coinciding with $f(z)$ in the $(n + 1)$ *st* roots of unity, then:

$$\lim_{n \to \infty} \int_{\mathbb{T}} |f(x) - p_n(x)|^2 |dx| = 0 \qquad \text{... (2.2)}$$

Furthermore, and as a consequence of (2.2) it holds too :

$$\lim_{n \to \infty} p_n(z) = f(z) \qquad \text{... (2.3)}$$

uniformly on compact subset of $\mathbb{D}$.

Throughout this Section we will assume that f is a continuous function on $\mathbb{T}$, and we will try to approximate f by sequences of interpolating Laurent polynomials on $\mathbb{T}$. This fact is basically motivated by the following,

Theorem 2.1 — [3] *Let C be an arbitrary Jordan curve of the finite z - plane. Then, any function f(z) continuous on C can be uniformly approximated on C by the sum of a polynomial in z and a polynomial in z.*

As an immediate consequence, we have

Corollary 2.2 — If f is any continuous function on $\mathbb{T}$, then it can be uniformly approximated on $\mathbb{T}$ by Laurent polynomials (L-polynomials).

In order to construct sequences of interpolant L-polynomials and as usual, it is crucial to make an adequate selection of the interpolation nodes. Concerning L_2 - *convergence*, a first result can be directly deduced from Erdös-Turan Theorem. Indeed, let $f(x)$ be a continuous function on $\mathbb{T}$. Set

$$f(x) = f(e^{i\theta}) = f_1(\theta) + if_2(\theta) \qquad \text{... (2.4)}$$

with $f_j \in C_{2\pi}$ $j = 1, 2$. Assume $\theta_{j,n} = \frac{2\pi j}{2n+1}, j = 1, ..., 2n+1$ and define $x_{j,n} = e^{i\theta_{j,n}}$, $j = 1, ..., 2n + 1$. Consider $t_{j,n}(\theta), j = 1, 2$ the corresponding trigonometric polynomials of order n interpolating $f_j(\theta)$ at the nodes $\theta_{k,n}, j = 1, 2, k = 1, ..., 2n+1$.

Define $T_n(x) = T_n(e^{i\theta}) = t_{n,1}(\theta) + i\, t_{n,2}(\theta)$ so that $T_n \in \Lambda_{-n,n}$ and clearly satisfies $T_n(x_{j,n}) = f(x_{j,n})$, $n = 1, ..., 2n + 1$. By virtue of unicity, $T_n(x)$ coincides with the unique L-polynomial in $\Lambda_{-n,n}$ interpolating $f(x)$ at the nodes $x_{j,n}, n = 1, ..., 2n+1$. Furthermore, it results:

$$\int_{\mathbb{T}} |f(x) - T_n(x)|^2 |dx| = \int_0^{2\pi} |f_1(\theta) - t_{n,1}(\theta)|^2 d\theta + \int_0^{2\pi} |f_2(\theta) - t_{n,2}(\theta)|^2 d\theta \qquad \text{... (2.5)}$$

and consequently $$\lim_{n \to \infty} \int_{\mathit{\Pi}} |f(x) - T_n(x)|^2 \, |dx| = 0. \quad \text{... (2.6)}$$

Result (2.6) will be generalized in different directions. Namely, other families of nodes on $\mathit{\Pi}$ will be considered, more general spaces of L-polynomials than $\Lambda_{-n,n}$ used and under appropriate conditions on function f, a result for the uniform norm on $\mathit{\Pi}$ deduced.

Thus, let $\{p(n)\}$ and $\{q(n)\}$ be two nondecreasing sequences of nonnegative integers such that $p(n) + q(n) = n$, $n = 1, 2, \ldots$ Assume that

$$\lim_{n \to \infty} p(n) = \infty \text{ and } \lim_{n \to \infty} q(n) = \infty \quad \text{... (2.7)}$$

Take $\{w_n\}$ a sequence of unimodular complex numbers, i.e. $w_n \in \mathit{\Pi}$, $n = 1, 2, \ldots$ and define

$$x_{j,n} = \sqrt[n+1]{w_n}, \; j = 1, \ldots, n+1 \quad \text{... (2.8)}$$

Consider, the unique interpolant $R_n \in \Lambda_{-p(n), q(n)}$ satisfying

$$R_n(x_{j,n}) = f(x_{j,n}), j = 1, \ldots, n+1 \quad \text{... (2.9)}$$

Then, we have

Theorem 2.3 — *In the above conditions, it holds*

$$\lim_{n \to \infty} \int_{\mathit{\Pi}} |f(x) - R_n(x)|^2 \, |dx| = 0$$

PROOF : We know that for $R_n(x)$, the following representation holds

$$R_n(x) = \sum_{j=1}^{n+1} L_{j,n}(x) f(x_{j,n}) \quad \text{... (2.10)}$$

where,

$$L_{j,n}(x) = \frac{x_{j,n}^{p+1}}{x^p} \frac{x^{n+1} - w_n}{(n+1)(x - x_{j,n})} j = 1, \ldots, n+1 \quad \text{... (2.11)}$$

(for the sake of simplicity, sometimes we will write p and q instead of $p(n)$ and $q(n)$).

Let $T_n(x)$ denote the L-polynomial in $\Lambda_{-p,q}$ of the best uniform approximation to $f(z)$ on $\mathit{\Pi}$. We introduce the notation

$$r_n(x) = f(x) - T_n(x) \text{ and } E_n(f) = \max_{x \in \mathit{\Pi}} |r_n(x)| \quad \text{... (2.12)}$$

Denote by $P_n(x)$ the L-polynomial in $\Lambda_{-p,q}$ interpolating $r_n(x)$ at the nodes $\{x_{j,n}\}_{j=1}^{n+1}$

We have

$$P_n(x) = R_n(x) - T_n(x)$$

and therefore,

$$\int_{\Pi} |f(x) - R_n(x)|^2 |dx| = \int_{\Pi} |r_n(x) - P_n(x)|^2 |dx|$$

$$\leq 2 \int_{\Pi} |r_n(x)|^2 |dx| + 2 \int_{\Pi} |P_n(x)|^2 |dx| = I_n' + I_n''$$

Clearly, $I_n' \leq 4\pi E_n^2(f)$. As for I_n'' we have

$$I_n'' = 2 \int_{\Pi} |P_n(x)|^2 |dx| = 2 \int_{\Pi} \left| \sum_{j=1}^{n+1} L_{j,n}(x) \, r_n(x_{j,n}) \right|^2 |dx|$$

$$\leq 2 \sum_{j=1}^{n+1} \sum_{k=1}^{n+1} |r_n(x_{j,n}) \overline{r_n(x_{k,n})}| \left| \int_{\Pi} L_{j,n}(x) \overline{L_{k,n}(x)} |dx| \right|$$

$$\leq 2 E_n^2(f) \sum_{j=1}^{n+1} \sum_{k=1}^{n+1} \left| \int_{\Pi} L_{j,n}(x) \overline{L_{k,n}(x)} |dx| \right|.$$

Now, by Residue Theorem, one has

$$\int_{\Pi} L_j(x) \overline{L_k(x)} |dx| = \int_{\Pi} \frac{x_{j,n}^{p+1} (x^{n+1} - w_n)}{x^p (n+1)(x - x_{j,n})} \frac{x_{k,n}^{-(p+1)} (x^{-(n+1)} - w_n^{-1})}{x^{-p} (n+1)(x^{-1} - x_{k,n}^{-1})} |dx|$$

$$= \frac{1}{i(n+2)^2} \frac{x_{j,n}^{p+1}}{x_{k,n}^{p} w_n)} \int_{\Pi} \frac{(x^{n+1} - w_n)}{(x - x_{j,n})} \frac{(x^{n+1} - w_n)}{(x - x_{k,n})} \frac{dx}{x^{n+1}}$$

$$= \frac{2\pi x_{j,n}^{p+1}}{(n+1)^2 x_{k,n}^{p} w_n} Res\,(x = 0).$$

Since,

$$\frac{x^{n+1} - w_n}{x - x_{j,n}} = \frac{x^{n+1} - x_{j,n}^{x+1}}{x - x_{j,n}} = x^n + x_{j,n} x^{n-1} + \ldots + x_{j,n}^{n-1} x + x_{j,n}^n;$$

Then,

$$Res\ (x = 0) = x_{k,n}^n + x_{j,n} x_{k,n}^{n-1} + \ldots + x_{j,n}^{n-1} x_{k,n} + x_{j,n}^n$$

If $j \neq k$, then $x_{j,n} \neq x_{k,n}$ and it results

$$Res\,(x=0) = \frac{x_{k,n}^n - x_{j,n}^n \dfrac{x_{j,n}}{x_{k,n}}}{1 - \dfrac{x_{j,n}}{x_{k,n}}} = \frac{x_{k,n}^{n+1} - x_{j,n}^{n+1}}{x_{k,n} - x_{j,n}} = 0.$$

If $j = k$, $Res\ (x = 0) = (n+1)\, x_{j,n}^n = \dfrac{(n+1)\, w_n}{x_{j,n}}$.

Thus,

$$\int_{\Pi} |L_j(x)|^2 |dx| = \frac{2\pi}{n+1};\ j = 1, \ldots, n+1.$$

Therefore, $I_n'' \leq 4\pi E_n^2(f)$. So, we finally have

$$\int_{\Pi} |f(x) - R_n(x)|^2 |dx| \leq 8\pi E_n^2(f). \qquad \ldots (2.13)$$

Since we know that $\lim_{n \to \infty} E_n(f) = 0$ (see Corollary (2.2)), proof follows. ■

Actually, taking the $(n+1)$ st roots of w_n ($|w_n| = 1$) instead of the roots of unity, does not represent a relevant generalization with respect to the selection of the interpolation nodes. In fact, this will be done by taking the zeros of systems of polynomials satisfying certain orthogonality properties on the unit circle Π. Recall that in the case of algebraic polynomials and functions defined on intervals of the real line, there are a lot of systems of interpolation nodes based on roots of complete orthonormal systems thoroughly investigated. (See e.g. [4] for a survey and references therein found).

Thus, let $\omega(\theta)$ be a weight function on $[0, 2\pi]$ such that

$$\int_0^{2\pi} \omega(\theta) d\theta = 1$$

(normalization condition). This enables us to introduce the following Hermitian product

$$\langle f, g \rangle_\omega = \int_0^{2\pi} f(e^{i\theta})\, \overline{g(e^{i\theta})}\, \omega(\theta)\, d\theta$$

By applying the Gram-Schmidt orthogonalization process to $\{1, x, \ldots, x^n\}$ an orthogonal basis $\{\rho_k(x)\}_{k=0}^n$ of monic polynomials can be deduced. The sequence $\{\rho_k(x)\}_{k=0}^n$ represents the system of monic orthogonal polynomials (or Szegö polynomials) with respect to the weight function $\omega(\theta)$ on the unit circle, satisfying

$$\langle \rho_n, \rho_m \rangle_\omega = K_n \delta_{n,m}, K_n > 0$$

It is well known (see [5]) that for each n the zeros of ρ_n lie inside $\mathbb{D}$. So, they cannot be taken as interpolation nodes. In order to overcome this drawback. William B. Jones *at al.*[6] proved the following,

Theorem 2.4 — *Let $\{w_n\}$ be a sequence of complex numbers on $\mathbb{T}$ and define*

$$B_n(x, w_n) = \rho_n(x) + w_n \rho_n^*(x) \qquad \dots (2.14)$$

Then

(i) B_n has exactly n distinct zeros on $\mathbb{T}$: $x_{j,n}, j = 1, \dots, n$

(ii) There exist positive numbers $\lambda_{j,n} j = 1, \dots, n$ uniquely determined, such that

$$\int_0^{2\pi} p(x)\overline{q(x)}\,\omega(\theta)\,d\theta = \sum_{j=1}^{n} \lambda_{j,n} p(x_{j,n}) \overline{q(x_{j,n})}\; p, q \in \Pi_{n-1} \text{ with } x = e^{i\theta}.$$

Remark 2.5 : When taking $\omega(\theta) = \frac{1}{2\pi}$, then it is known that $\rho_n(x) = x^n$. Therefore, $B_n(x, w_n) = x^n + w_n$. Then, if we put in (2.14) $w_n = -1$. The zeros $x_{j,n}$ are the $n - st$ roots of unity.

Polynomials $B_n(x, w_n)$ are used to be called para-orthogonal since they satisfy the following orthogonality conditions —

$$(B_n, z^k)_\omega = 0 \; 1 \le k \le n-1, (B_n, z^n)_\omega \ne 0, \; (B_n, 1)_\omega \ne 0.$$

Let $\{x_{j,n} : j = 1, \dots, n+1\}$ be the zeros of $B_{n+1}(x, w_{n+1}) = \rho_{n+1}(x) + w_{n+1} \rho_{n+1}^*(x)$, $\{w_n\}$ being an arbitrary sequence of unimofular complex numbers. Let f be a continuous function on $\mathbb{T}$ and $R_n \in \Lambda_{-p,q}$ the interpolant of f at the nodes $\{x_{j,n} : j = 1, \dots, n+1\}$ with p and q nonnegative integers such that $p + q = n$. We can write: $R_n(x) = \sum_{j=1}^{n+1} L_j(x) f(x_{j,n})$ where

$$L_j(x) = \frac{x_{j,n}^p B_{n+1}(x, w_{n+1})}{x^p B'_{n+1}(x_{j,n})(x - x_{j,n})} \; j = 1, \dots, n+1$$

and $L_j(x_{k,n}) = \delta_{j,k}$. Thus, by (ii) in Theorem (2.4), it follows

$$\int_{\mathbb{T}} L_j(x)\overline{L_k(x)}\,\omega(\theta)\,d\theta = \delta_{j,k} \lambda_{j,n+1}$$

with $\lambda_{j,n+1}; j = 1, \dots, n+1$ as given in Theorem 2.4. Therefore, proceeding as in the proof of Theorem 2.3, we can deduce

$$\int_{\Pi} |f(x)\, R_n(x)|^2\, \omega(\theta)\, d\theta \leq 4E_n^2(f), \qquad \ldots (2.15)$$

where $x = e^{i\theta}$ and with $E_n(f)$ given by (2.12). In other words, the following Theorem has been proved.

***Theorem* 2.6** — *Let f be a continuous function on Π and let $\{x_{j,n} = j = 1, ..., n+1\}$ be the zeros of $B_{n+1}(x, w_{n+1})$ as defined above. Let $R_n(x)$ be the interpolant of f in $\Lambda_{-p,q}$ at the nodes $\{x_{j,n}\}_{j=1}^{n+1}$ with $p = p(n)$ and $q = q(n)$ satisfying (2.7) and $p + q = n$. Then*

$$\lim_{n \to \infty} \int_{\Pi} |f(x) - R_n(x)|^2\, \omega(\theta)\, d\theta = 0.$$

Remark 2.7 : Theorem 2.6 was earlier proved by A. Bultheel et al. in [7] (see also [8]) in a more general framework rational interpolants with prescribed poles not on Π were used. Observe that L-polynomials are rational functions with poles at the origin and infinity. However, the proof presented here is more constructive in the sense that an error bound like (2.15) for the $L_{2,\omega}$-norm has been given.

Next, we are going to consider the uniform norm on Π, i.e.,

$$\|f\|_{\Pi} = \max_{x \in \Pi} |f(x)|$$

f being a continuous function on Π. Now, we will restrict ourselves to nodes $x_{j,n}\, j = 1, .., n+1$ which are the roots of w_n, with $|w_n| = 1$, $n = 1, 2, ...,$ i.e.

$$x_{j,n}^{n+1} = w_n \quad j = 1, ..., n+1.$$

Set $W_n(x) = x^{n+1} - w_n$, then one can write

$$R_n(x) = \sum_{j=1}^{n+1} L_{j,n}(x) f(x_{j,n}),$$

$R_n(x)$ being the unique interpolant in $\Lambda_{-p,q}$ satisfying

$$R_n(x_{j,n}) = f(x_{j,n})\; j = 1, ..., n+1$$

and where, as already known,

$$L_{j,n} = \frac{x_{j,n}^p\, W_n(x)}{x^p\, W_n'(x_{j,n}(x - x_{j,n})} \in \Lambda_{-p,q}, j = 1, .., n+1$$

Thus, for all $x \in \Pi$, one has

$$|L_{j,n}(x)|^2 = \frac{|W_n(x)|^2}{(n+1)^2\,|x - x_{j,n}|^2}$$

Therefore,

$$\sum_{j=1}^{n+1} |L_{j,n}(x)|^2 = \frac{|W_n(x)|^2}{(n+1)^2} \sum_{j=1}^{n+1} \frac{1}{|x - x_{j,n}|^2} \qquad \text{... (2.16)}$$

But, for all $x \in \mathit{\Pi}$,

$$|x - x_{j,n}|^2 = (x - x_{j,n})\,\overline{(x - x_{j,n})} = -\frac{(x - x_{j,n})^2}{x x_{j,n}}$$

So,

$$\sum_{j=1}^{n+1} \frac{1}{|x - x_{j,n}|^2} = -x \sum_{j=1}^{n+1} \frac{x_{j,n}}{(x - x_{j,n})^2} = x \sum_{j=1}^{n+1} \frac{(x - x_{j,n}) - x}{(x - x_{j,n})^2}$$

$$= x\left(\sum_{j=1}^{n+1} \frac{1}{(x - x_{j,n})} - x \sum_{j=1}^{n+1} \frac{1}{(x - x_{j,n})^2} \right)$$

On the other hand,

$$\sum_{j=1}^{n+1} \frac{1}{(x - x_{j,n})^2} = \frac{W_n'(x)}{W_n(x)} = \frac{(n+1)\,x^n}{x^{n+1} - w_n}$$

and

$$-\sum_{j=1}^{n+1} \frac{1}{(x - x_{j,n})^2} = (n+1)\,\frac{n x^{n-1}\,(x^{n+1} - w_n) - (n+1)\,x^{2n}}{(x^{n+1} - w_n)^2}$$

$$= -\frac{(n+1)\,x^{n-1}\,(x^{n+1} + n w_n)}{(x^{n+1} - w_n)^2}$$

Thus, one obtains

$$0 < \sum_{j=1}^{n+1} \frac{1}{|x - x_{j,n}|^2} = x\left(\frac{(n+1)\,x^n}{x^{n+1} - w_n} - x\,\frac{(n+1)\,x^{n-1}\,(x^{n+1} + n w_n)}{(x^{n+1} - w_n)^2} \right)$$

$$= -\frac{w_n\,(n+1)^2\,x^{n+1}}{(x^{n+1} - w_n)^2} = \frac{(n+1)^2}{|x^{n+1} - w_n|^2} = \frac{(n+1)^2}{|W_n(x)|^2} \qquad \text{... (2.17)}$$

Thus, by (2.16), one deduces that for all $x \in \Pi$ $\sum_{j=1}^{n+1} |L_j(x)|^2 = 1$.

Now we define

$$H_n(x) = \sum_{j=1}^{n+1} |L_j(x)|$$

where $x = e^{i\theta}$ and

$$\|H_n\|_{\Pi} = \max_{x \in \Pi} H_n(x) \quad \text{... (2.18)}$$

So, by applying Cauchy-Schwartz inequality, it follows, for all $x \in \Pi$,

$$H_n(x) = \sum_{j=1}^{n+1} |L_j(x)| \leq \left(\sum_{j=1}^{n+1} |L_j(x)|^2 \right)^{1/2} \left(\sum_{k=1}^{n+1} 1 \right)^{1/2}$$

Therefore, for all $x \in \Pi$,

$$H_n(x) \leq \sqrt{n+1}$$

which gives

$$\|H_n\|_{\Pi} \leq \sqrt{n+1}.$$

We summarize this result in the following :-

Lemma 2.8 — Let p and q be nonnegative integers such that $p + q = n$ and $L_{j,n}(x)\, j = 1, ..., n$ be the fundamental Lagrange Laurent polynomials in $\Lambda_{-p,q}$ for the nodes $x_{j,n}\, j = 1, ..., n+1, x_{j,n}$ being the $(n + 1) - st$ roots of w_n, with $|w_n| = 1, n = 1, 2, ...$ Then,

$$\|H_n\|_{\Pi} = \max_{x \in \Pi} \sum_{j=1}^{n+1} |L_j(x)| \leq \sqrt{n+1}, n = 1, 2$$

For given nondecreasing sequences $\{p(n)\}$ and $\{q(n)\}$ of nonnegative integers satisfying $p(n) + q(n) = n$ and (2.7), one knows that, (Corollary 2.2)

$$\lim_{n \to \infty} E_n(f) = \lim_{n \to \infty} \|f - T_n\|_{\Pi} = 0,$$

$T_n \in \Lambda_{-p(n), q(n)}$ being the best L-polynomial in $\Lambda_{-p(n), q(n)}$ with respect to the uniform norm. In order to check how fast $E_n(f)$ tends to zero as n tends to infinity, the following result is required:

Theorem 2.9 — *(Jackson's Theorem [9]) — For all* $f \in C_{2\pi}$, $\gamma_n(f) \leq \omega\left(f, \frac{\pi}{n+1}\right)$ *where* $\gamma_n(f) = \max_{x \in [0, 2\pi]} |f(\theta) - t_n(\theta)|$ $t_n(\theta)$ *being the best trigonometric polynomial of order n and where* ω *denotes the modulus of continuity of f.*

Now, we have immediately the following

Proposition 2.10 — Let f be a continuous function on Π and p and q nonnegative integers such that $p + q = n$. Then,

$$E_n(f) \le 2\omega\left(f, \frac{\pi}{s+1}\right)$$

where $s = \min(p, q)$ and $E_n(f)$ as given by (2.12).

PROOF : Write $f(x) = f(e^{i\theta}) = f_1(\theta) + \text{i}f_2(\theta); f_j \in C_{2\pi}, j = 1, 2$

By Theorem 2.9

$$\gamma_s(f_j) \le \omega\left(f_j, \frac{\pi}{s+1}\right), \; j = 1, 2, \; s = \min(p, q)$$

Here, $\gamma_s(f_j) = \max_{x \in [0, 2\pi]} |f_j(\theta) - t_j(\theta)| = \|f_j - t_j\|_{[0, 2\pi]}, j = 1, 2, t_j$ being the best trigonometric polynomial of order s at most in the uniform norm to $f_j, j = 1, 2 = 1, 2$.

Set $L(x) = L(e^{i\theta}) = t_1(\theta) + it_2(\theta) \in \Lambda_{-s,s} \subset \Lambda_{-p,q}$. Thus,

$$E_n(f) \le \|f - L\|_{\Pi} \le \|f_1 - t_1\|_{[0, 2\pi]} + \|f_2 - t_2\|_{[0, 2\pi]}$$

$$\le \omega\left(f_1, \frac{\pi}{s+1}\right) + \omega\left(f_2, \frac{\pi}{s+1}\right) \le 2\omega\left(f, \frac{\pi}{s+1}\right) \qquad \blacksquare$$

We are now in a position to state the main result of this Section.

Theorem 2.11 — *Let f be a continuous function on Π and {p(n)} and {q(n)} sequences of nonnegative integers with $p(n) + q(n) = n$, $n = 1, 2, \ldots$ and satisfying*

$$\lim_{n \to \infty} \frac{p(n)}{n} = r; \; 0 < r < 1.$$

Assume that $\omega(f, \delta) = O(\delta^p); p > 1/2$ and $\delta \to 0$. Denote by $R_n(x) = R_n(f, x)$ the interpolant of f in $\Lambda_{-p(n), q(n)}$ at the nodes $x_{j,n}, j = 1, \ldots, n+1$ the (n + 1) st roots of w_n, with $|w_n| = 1$, n = 1, 2, ... Then,

$$\lim_{n \to \infty} R_n(f, x) = f(x)$$

uniformly on Π.

PROOF : Let T_n be the best L-polynomial to f in $\Lambda_{-p(n), q(n)}$ with respect to the uniform norm. One has, for all $x \in \Pi$,

$$f(x) - R_n(f, x) = f(x) - T_n(x) + T_n(x) - R_n(f, x)$$

Since $T_n \in \Lambda_{-p(n), q(n)}$, it holds $R_n(T_n, x) = T_n(x)$. Therefore,

$$f(x) - R_n(f, x) = f(x) - T_n(x) - R_n(f - T_n, x)$$

Recall : $R_n(f - T_n, x) = \sum_{k=1}^{n+1} L_k(x)\,(f(x_{k,n}) - T_n(x_{k,n}))$. Thus, for all $x \in \Pi$,

$$|f(x) - R_n(f, x)| \le E_n(f) + \sum_{k=1}^{n+1} |L_k(x)|\,|f(x_{k,n}) - T_n(x_{k,n})|$$

$$\le E_n(f)\left(1 + \sum_{k=1}^{n+1} |L_k(x)|\right)$$

By Lemma 2.8, and Proposition 2.10, it follows, for all $x \in \Pi$,

$$|f(x) - R_n(f, x)| \le E_n(f)\,(1 + \sqrt{n+1}) \le 2\omega\left(f, \frac{\pi}{s(n)+1}\right)(1 + \sqrt{n+1}) \quad \text{... (2.19)}$$

Here $s(n) = \min\,(p(n), q(n))$. It can be checked that

$$\lim_{n \to \infty} \frac{s(n)}{n} = \begin{cases} r & \text{if } 0 < r < 1/2 \\ 1 - r & \text{if } 1/2 < r < 1 \end{cases} \quad \text{... (2.20)}$$

Thus, by (2.19) and (2.20) proof easily follows. ■

Remark 2.12 — Let X be an infinite triangular array of nodes in $[-1, 1]$, i.e.,

$$X = \{x_{j,n} : 1 \le j \le n, n = 1, 2, \ldots\} \subset [-1, 1].$$

As is well known, $\lambda_{k+1}(X, x) = \sum_{j=1}^{k+1} |l_{j,k+1}(X, x)|$ is called the Lebesgue function of order $k + 1$ of X, where $l_{j,k+1}$ are the fundamental Lagrange polynomials for the nodes $x_1^{(k+1)}, x_2^{(k+1)}, \ldots, x_{k+1}^{(k+1)}$. The quantity: $\lambda_{k+1}(X) = \max_{x \in [-1,1]} \lambda_{k+1}(X, x)$ is called the Lebesgue consant of order $k + 1$ of X.

Let T be the array whose kth row is $\xi_1^{(k)}, \ldots, \xi_k^{(k)}$, i.e., the zeros of the Chebyshev polynomial of the first kind $T_k(x)$. Then [10]

$$\lambda_{k+1}(T) \le \frac{2}{\pi} \log k + 1, \quad k = 1, 2, \ldots$$

Thus, observe that $\|H_n\|_{\Pi}$, as defined by (2.18), plays the same role as the Lebesgue constant. A first question is whether for the roots of unity, a sharper upper bound than $\sqrt{n+1}$ can be obtained.

On the other hand and since the roots of any unimodular complex number are the zeros of para-orthogonal polynomials with respect to the Lebesgue measure, we could also consider other systems of interpolation nodes as zeros of para-orthogonal polynomials $B_n(x, w_n)$ ($|w_n| = 1$) as given by (2.14), with respect to another weight function. For this aim, set $\omega(\theta) = \dfrac{1 - r^2}{2\pi(1 - 2\cos\theta + r^2)}$, $0 \le r < 1$, Poisson measure). Note that for $r = 0$ we have the Lebesgue measure. In order to estimate

$\| H_n \|_{\varPi}$, we have computed $H_n(x)$ for several values of $x \in \varPi$. Table I shows the results. If we apply Cauchy-Schwartz inequality to $H_n(x)$ as we have done before, we have that for all $x \in \varPi$,

$$H_n(x) = \sum_{j=1}^{n+1} | L_j(x) | \le \left(\sum_{j=1}^{n+1} | L_j(x) |^2 \right)^{1/2} \left(\sum_{k=1}^{n+1} 1 \right)^{1/2}$$

We have also calculated $\sum_{j=1}^{n+1} | L_j(x) |^2$ for several $x \in \varPi$. Results are displayed in Table II.

TABLE I

$r = 0.5, w_n = 1$	$n = 3$	$n = 5$	$n = 7$	$n = 9$	$n = 11$
$x = 1$	1.397784	1.57992	1.723141	1.910335	1.9392
$x = -1$	2.449491	2.373997	2.933358	3.029046	3.208695
$x = e^{\frac{\pi}{2}i}$	1.962709	1.682776	2.397946	1.8378	2.645338
$x = e^{\frac{\pi}{4}i}$	1.430733	1.933975	1.948278	1.359033	1.706982
$x = e^{\frac{\pi}{6}i}$	1.056523	1.377201	1.803354	2.109312	1.144155
$x = e^{\frac{\pi}{3}i}$	1.817217	1.933035	0.0372543	2.316734	2.323518
$x = e^{\frac{\pi}{5}i}$	1.127013	1.657399	2.002495	2.039703	1.573344
$x = e^{\frac{3\pi}{10}i}$	1.683758	2.00178	1.483753	1.756832	2.472193
$x = e^{\frac{\pi}{7}i}$	1.143668	1.141888	1.560843	1.955481	2.11586
$x = e^{\frac{5\pi}{14}i}$	1.89301	1.807811	1.441098	2.515929	1.893633

TABLE II

$r = 0.5, w_n = 1$	$n = 3$	$n = 5$	$n = 7$	$n = 9$	$n = 11$
$x = 1$	0.6969723	0.7441738	0.7819141	0.822225	0.8336556
$x = -1$	1.545455	1.383746	1.297395	1.218227	1.205489
$x = e^{\frac{\pi}{2}i}$	1.121215	1.154252	1.091594	1.09958	1.069524
$x = e^{\frac{\pi}{4}i}$	1.16407	1.090565	0.9031604	0.9434807	1.083503
$x = e^{\frac{\pi}{6}i}$	0.96376	1.080369	1.051778	0.9604875	0.8898444
$x = e^{\frac{\pi}{3}i}$	1.272731	0.9463865	0.001387876	1.163986	0.9842566
$x = e^{\frac{\pi}{5}i}$	1.049032	1.11499	0.997557	0.8971838	0.9199576
$x = e^{\frac{3\pi}{10}i}$	1.244905	1.004661	0.9151806	1.110349	1.078075
$x = e^{\frac{\pi}{7}i}$	0.9034595	1.03289	1.05977	1.012111	0.932241
$x = e^{\frac{5\pi}{14}i}$	1.278674	0.918635	1.079651	1.14263	0.9504793

From Tables I and II it seems that could be conjectured a positive constant K there exists such that

$$\| H_n \|_{\mathbb{T}} \leq k \sqrt{n+1}$$

However, we believe that a deeper investigation is required.

3. Interpolation of Analytic Functions

We will first functions f analytic in $\mathbb{D}$ and continuous on $\mathbb{D} \cup \mathbb{T}$. As is well known, $f(z)$ can be uniformly approximated by polynomials in compact subsets of $\mathbb{D}$.

In order to deduce the uniform convergence in compacts of $\mathbb{D}$ of certain sequences of interpolating polynomials with nodes on $\mathbb{T}$, the following results are previously required.

Lemma 3.1 — In the above conditions on the function $f(z)$, let $T_{n,p}(x)$ be the best Laurent polynomial to $f(z)$ in $\Lambda_{-p,n}$ with respect to the uniform norm, p and n being nonnegative integers. Then,

$$\lim_{n \to \infty} \| f - T_{n,p} \|_{\mathbb{T}} = 0, \ (p \text{ fixed})$$

Proof : Set $g(x) = x^p f(x)$. Obviously g is an analytic function in $\mathbb{D}$ and continuous on $\mathbb{D} \cup \mathbb{T}$.

Let P_n be the polynomial in Π_{n+p} of best approximation to g in the uniform norm. Define

$$R_n(x) = x^{-p} P_n(x) \in \Lambda_{-p,n}.$$

Thus, $\| f - T_{n,p} \|_{\mathbb{T}} \leq \| f - R_n \|_{\mathbb{T}} = \| f - x^{-p} P_n(x) \|_{\mathbb{T}} = \| g - P_n(x) \|_{\mathbb{T}} \to 0$ as n tend to infinity. ■

Proceeding as in Theorem 2.6 and using Lemma 3.1 we can also deduce the following :

Proposition 3.2 — Let f be a function analytic in $\mathbb{D}$ and continuous on $\mathbb{D} \cup \mathbb{T}$. Let $\{x_{j,n}\}_{j=1}^{n+1}$ be the zeros of $B_{n+1}(x, w_{n+1})$ as given in (2.14). Let $R_n(x)$ be the interpolant of f in $\Lambda_{-p(n),q(n)}$ at the nodes $\{x_{j,n}\}_{j=1}^{n+1}$ $\{p(n)\}$ and $\{q(n)\}$ being nondecreasing sequences of nonnegative integers such that $p(n) + q(n) = n$ and either $\lim_{n \to \infty} p(n) = \infty$ or $\lim_{n \to \infty} q(n) = \infty$. Then, $\lim_{n \to \infty} \int_{\mathbb{T}} |f(x) - R_n(x)|^2 \omega(\theta)\, d\theta = 0$ where $\omega(\theta)$ is a weight function on $[0, 2\pi]$ generating polynomials $B_{n+1}(x, w_{n+1})$.

Now, we can prove the following :

***Theorem* 3.3** — *Let $\omega(\theta)$ be a weight function on $[0, 2\pi]$ sch that $\int_0^{2\pi} \frac{d\theta}{\omega(\theta)} < \infty$. Let $\{w_n\}$ be a sequence of complex numbers on $\mathbb{T}$ and set*

$$B_n(z, w_n) = \rho_n(z) + w_n \rho_n^*(z) \quad n = 1, 2, \dots$$

Let $\{x_{j,n+1}\}_{j=1}^{n+1}$ *be the zeros of* $B_{n+1}(z, w_{n+1})$ *and consider* $P_n \in \Pi_n$ *interpolating* f *at those nodes. Then :*

$$\lim_{n \to \infty} P_n(z) = f(z)$$

uniformly on compact subsets of $\mathbb{D}$.

PROOF : By the Cauchy integral formula, it holds

$$f(z) - P_n(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(t) - P_n(t)}{t - z} dt, \; z \in \mathbb{D}$$

Let K be a compact in $\mathbb{D}$ so that dist $(K, \mathbb{T}) = \delta > 0$. Thus, for all $t \in K$ and for all $z \in \mathbb{T}$, $|t - z| \geq \delta$.

Hence,

$$|f(z) - P_n(z)| \leq \frac{1}{2\pi} \int_{\mathbb{T}} \frac{|f(t) - P_n(t)|}{|t - z|} |dt|$$

$$\leq \frac{1}{2\pi\delta} \int_{\mathbb{T}} |f(t) - P_n(t)| |dt| = \frac{1}{2\pi\delta} \int_{\mathbb{T}} |f(t) - P_n(t)| \frac{\sqrt{\omega(\theta)}}{\sqrt{\omega(\theta)}} |dt|$$

Making use of Cauchy-Schwarz inequality for integrals, it follows,

$$|f(z) - P_n(z)| \leq \frac{1}{2\pi\delta} \left(\int_{\mathbb{T}} |f(t) - P_n(t)|^2 \omega(\theta) |dt| \right)^{1/2} \left(\int_0^{2\pi} \frac{d\theta}{\omega(\theta)} \right)^{1/2}$$

Now, proof follows by Proposition 3.2, when taking $p = 0$.

Remark 3.4 : When taking $\omega(\theta) = \frac{1}{2\pi}$ we have already seen that $B_n(z, w_n) = z^n + w_n$. Thus, setting $w_n = -1$, for each n, Theorem 1 in [2] is recovered.

Assume now that $f(z)$ is analytic in a region $\mathbb{B}$ such that $\mathbb{T} \subset \mathbb{B}$, C being its boundary. Let $\{x_k\}_{k=1}^{n+1}$ be $(n + 1)$ distinct nodes on $\mathbb{T}$ and let R_n be the Laurent polynomial in $\Lambda_{-p,q}$ $(p + q = n, p \geq 0, q \geq 0)$ interpolating f at these nodes. Making use of Hermite formula for the remainder in polynomial interpolation[3], one can write, for all $z \in \mathbb{B}, z \neq 0$,

$$f(z) - R_n(z) = \frac{1}{2\pi i} \int_C \left(\frac{t}{z} \right)^p \frac{(z - x_1) \dots (z - x_{n+1})}{(t - x_1) \dots (t - x_{n+1})} \frac{f(t)}{t - z} dt. \qquad \dots (3.1)$$

Nodes $\{x_k\}_{k=1}^{n+1}$ are initially going to be chosen as the $(n+1)$ roots of $w_n \in \mathbb{T}$, i.e., the zeros of

$$Q_{n+1}(z) = (z - x_1) \dots (z - x_{n+1}) = z^{n+1} - w_n.$$

Thus, (3.1) becomes

$$f(z) - R_n(z) = \frac{1}{2\pi i} \int_C \left(\frac{t}{z}\right)^p \frac{z^{n+1} - w_n}{t^{n+1} - w_n} \frac{f(t)}{t - z} dt.$$

Assume $C = C_r \cup C_R$, where

$C_r = \{z : |z| = r\}$ and $C_R = \{z ; |z| = R\}$

with $0 < r < 1 < R$. Thus,

$$f(z) - R_n(z) = \frac{1}{2\pi i} \int_{C_r} \left(\frac{t}{z}\right)^p \frac{z^{n+1} - w_n}{t^{n+1} - w_n} \frac{f(t)}{t - z} dt$$

$$+ \frac{1}{2\pi i} \int_{C_R} \left(\frac{t}{z}\right)^p \frac{z^{n+1} - w_n}{t^{n+1} - w_n} \frac{f(t)}{t - z} dt \qquad \dots (3.2)$$

$$= I_1 + I_2$$

As indicated in Section 1, we are mainly interested in evaluating the error on the unit circle $\mathbb{T}$, i.e. $|z| = 1$.

So, we have

$$|I_1| \le \frac{2r^{p+1}}{1 - r^{n+1}} \frac{M_r(f)}{1 - r}, M_r(f) = \max_{t \in C_r} \{|f(z)|\} \qquad \dots (3.3)$$

and

$$|I_2| \le \frac{2R^{p+1}}{1 - R^{n+1}} \frac{M_R(f)}{R - 1}, M_R(f) = \max_{t \in C_R} \{|f(z)|\}. \qquad \dots (3.4)$$

By (3.2), (3.3), (3.4), the following Theorem has been proved :

Theorem 3.5 — *Let f be an analytic function on the annulus $\{z : r < |z| < R\}$ with $0 < r < 1 < R$. Let $\{p(n)\}$ and $\{q(n)\}$ be sequences of nonnegative integers such that $p(n) + q(n) = n$ and*

$$\lim_{n \to \infty} p(n) = \lim_{n \to \infty} q(n) = \infty.$$

Let $\{w_n\}$ be a given sequence of complex numbers on $\mathbb{T}$ and $R_n(z) = R_n(f, z)$ the L-polynomial *in* $\Lambda_{-p(n),\,q(n)}$ *interpolating* $f(z)$ *at the nodes* $\left\{x_{j,n}\right\}_{j=1}^{n+1}$ *where*

$$x_{j,n}^{n+1} = w_n, j = 1, \ldots, n+1$$

Then, the sequence $\{R_n(f, z)\}$ *converges to* $f(z)$ *uniformly on* $I\!\!T$.

Example — Take $f(z) = e^z$, then we can make $r = 0$ and R any arbitrary positive number greater than one. Then, for all $z \in I\!\!T$,

$$|f(z) - R_n(f, z)| \leq \frac{2M_R(f)\, R^{p+1}}{(R^{n+1} - 1)(R - 1)}$$

$$= \frac{2e^R R^{p+1}}{(R^{n+1} - 1)(R - 1)} = F_n(R) \qquad \ldots (3.5)$$

Since, $\lim_{R \to 1} + F_n(R) = \lim_{R \to \infty} F_n(R) = \infty$. We can write,

$$|f(z) - R_n(f, z)| \leq \gamma_n, z \in I\!\!T$$

where

$$\gamma_n = \min_{R > 1} F_n(R).$$

A similar result can be obtained for quasi-Hermite interpolation. Indeed, as before, let $f(z)$ be an analytic function on a region $I\!\!B$ such that $I\!\!T \subset I\!\!B$ and $C = \delta I\!\!B$. Let $\left\{x_j\right\}_{j=1}^{n+1}$ be istinct points on $I\!\!T$. We know that there exists a unique interpolant in $\Lambda_{-n,n}$ satisfying the following quasi-Hermite interpolation conditions

$$H_{2n}(x_j) = f(x_j), j = 1, \ldots, n+1$$

and

$$H'_{2n}(x_j) = f'(x_j), j = 1, \ldots, n. \qquad \ldots (3.6)$$

Furthermore, it holds, for all $z \in I\!\!B, z \neq 0$,

$$f(z) - H_{2n}(z) = \frac{1}{2\pi i} \int_C \left(\frac{t}{z}\right)^n \frac{(z - x_1)^2 \ldots (z - x_n)^2 (z - x_{n+1})}{(t - x_1)^2 \ldots (t - x_n)^2 (t - x_{n+1})} \frac{f(t)}{t - z}\, dt \qquad \ldots (3.7)$$

Setting, $C = C_r \cup C_R$ where $C_r = \{z ; |z| = r\}$, $C_R = \{z : |z| = R\}$ and taking $\left\{x_j\right\}_{j=1}^{n+1}$ as the $(n + 1) - st$ roots of $w_n, w_n \in I\!\!T$, we can write

$$f(z) - H_{2n}(z) = \frac{1}{2\pi i} \int_{C_r} \left(\frac{t}{2}\right)^n \frac{(z^{n+1} - w_n)^2}{(z - x_{n+1})} \frac{f(t)}{t - z} dt$$

$$+ \frac{1}{2\pi i} \int_{C_R} \left(\frac{t}{z}\right)^n \frac{(z^{n+1} - w_n)^2}{(z - x_{n+1})} \frac{(t - x_{n+1})}{(t^{n+1} - \omega_n)^2} \frac{f(t)}{t - z} dt \quad \text{... (3.8)}$$

For any $z \in \mathit{IT}$, it holds

$$| I_1 | \leq 2M_r(f) \left(\frac{1 = r}{1 - r}\right) \frac{(n+1)\, r^{n+1}}{(1 - r^{n+2})^2} \quad \text{... (3.9)}$$

and

$$| I_1 | \leq 2M_R(f) \left(\frac{1 + R}{1 - R}\right) \frac{(n+1)\, R^{n+1}}{(1 - R^{n+1})^2} \quad \text{... (3.10)}$$

with $M_r(f)$ and $M_R(f)$ as given by (3.3) and (3.4) respectively. Thus, by By (3.8), (3.9) and (3.10), it follows

***Theorem* 3.6** — *In the above conditions, the sequence of quasi-Hermite interpolants $H_{2n} \in \Lambda_{-n,n}$ converges to $f(z)$ uniformly on IT.*

Remark 3.7 : The reason for considering these quasi-Hermite interpolants is the following. Let $I_{n+1} \sum_{j=1}^{n+1} \lambda_j f(x_j)$ be the $(n + 1)$ - point quadrature formula as defined in Theorem 2.4, (Szegö quadrature formula). Then, for any analytic function on a region IB containing IT, it holds[12],

$$\int_0^{2\pi} f(x)\, \omega(\theta)\, d\theta - I_{n+1}(f) = \int_0^{2\pi} (f(x) - H_n(x))\, \omega(\theta)\, d\theta, x = e^{i\theta}$$

Let us next consider more general choices of interpolation nodes on IT. As in previous Section 2, these will be zeros of para-orthogonal polynomials. Thus, let $\omega(\theta)$ be a weight function on $[0, 2\pi]$ and $\{w_n\}$ a given sequence of complex numbers on IT. Let $\{\varphi_n\}$ be the sequence of orthonormal Szegö polynomials and set

$$\chi_n(z) = \chi_n(z, ow_n) = \varphi_n(z) + w_n \varphi_n^*(z)$$

The following result will be needed later, (see [11]),

***Theorem* 3.8** — *In the above conditions, the following holds :*

(1) $\lim_{n \to \infty} | \chi_n(z) |^{1/n} = | z |$ uniformly on any compact $K \subset \mathit{IE}$.

(2) $\lim_{n \to \infty} | \chi_n(z) |^{1/n} = 1$ uniformly on any compact $K \in \mathit{ID}$.

(3) Set $M_n = \max_{z \in \Pi} |\chi_n(z)|$, then $\lim_{n \to \infty} M_n^{1/n} = 1$.

Let $\{p(n)\}$ and $\{q(n)\}$ be two arbitrary sequences of nonnegative integers such that $p(n) + q(n) = n$ and $\lim_{n \to \infty} \frac{p(n)}{n} = 2, 0 < s < 1$. Let $\{x_j\}_{j=1}^{n+1}$ be the zeros of $\chi_{n+1}(z)$ as in Theorem 3.8. For each n, let $R_n(z)$ denote the interpolant to $f(z)$ in $\Lambda_{-p(n), q(n)}$ at the nodes $\{x_j\}_{j=1}^{n+1}$. Then, one has,

Theorem 3.9 — *Let $f(z)$ be an analytic function in the annulus $\{z : r < |z| < R\}$ with $0 < r < 1 < R$.*

Then

$$\limsup_{n \to \infty} |f(z) - R_n(z)|^{\frac{1}{n+1}} \le \lambda < 1$$

uniformly on Π, where $\lambda = \max\left(r^s, \frac{1}{R^{1-s}}\right)$.

PROOF : By (3.1), one has, for all $z \in \Pi$

$$f(z) - R_n(z) = \frac{1}{2\pi i} \int_{C_r} \left(\frac{t}{2}\right)^p \frac{\chi_{n+1}(z)}{\chi_{n+1}(t)} \frac{f(t)}{t-z} dt$$

$$\frac{1}{2\pi i} \int_{C_R} \left(\frac{t}{2}\right)^p \frac{\chi_{n+1}(z)}{\chi_{n+1}(t)} \frac{f(t)}{t-z} dt \qquad \text{... (3.11)}$$

$$= I_1 + I_2$$

Then,

$$|f(z) - R_n(z)| \le |I_1| + |I_2| \qquad \text{... (3.12)}$$

Setting $M_r(f) = \max_{t \in C_r} \{|f(z)|\}$, $M_R(f) = \max_{t \in C_R} \{|f(z)|\}$, $M_n = \max_{z \in \Pi} |\chi_n(z)|$, we have

$$|I_1| \le \frac{M_r(f)}{1-r} r^{p(n)+1} M_n \max_{t \in C_r} |g_n(z)|$$

where $g_n(z) = \frac{1}{\chi_{n+1}(z)}$ which is continuous on C_r. Thus

$$|I_1|^{\frac{1}{n+1}} \le \left(\frac{M_r(f)}{1-r}\right)^{\frac{1}{n+1}} r^{\frac{p(n)+1)}{n+1}} M_n^{\frac{1}{n+1}} \left(\max_{t \in C_r} |g_n(z)|\right)^{\frac{1}{n+1}}$$

Since $g_n(z)$ is a continuous function on C_r, $\max_{t\in C_r} |g_n(t)| = |g_n(t_n)|$, $t_n \in C_r$. Thus,

$$\left(\max_{t\in C_r} |g_n(t)|\right)^{1/n} = |g_n(t_n)|^{1/n} \leq \max_{t\in C_r} |g_n(t)|^{1/n},$$

By (3) in Theorem 3.8, $\lim_{n\to\infty} M_n^{1/n} = 1$ and therefore,

$$\limsup_{n\to\infty} |I_1|^{\frac{1}{n+1}} \leq r^s \limsup_{n\to\infty} \left(\max_{t\in C_r} |g_n(t)|^{1/(n+1)} \right)$$

$$\leq r^s \max_{t\in C_r} \limsup_{n\to\infty} |g_n(t)|^{1/(n+1)}$$

$$= r^s \max_{t\in C_r} \limsup_{n\to\infty} \left| \frac{1}{\chi_{n+1}(t)} \right|^{1/(n+1)}$$

$$\leq r^s \max_{t\in C_r} \left\{ \frac{1}{\liminf_{n\to\infty} |\chi_{n+1}(t)|^{1/(n+1)}} \right\}$$

Hence, by (2) in Theorem 3.8 it follows,

$$\limsup_{n\to\infty} |I_1|^{\frac{1}{n+1}} \leq r^s < 1.$$

Similarly, by (1) in Theorem 3.8 it can be deduced,

$$\limsup_{n\to\infty} |I_2|^{\frac{1}{n+1}} \leq \left(\frac{1}{R}\right)^{1-s} < 1$$

Now, by (3.12), we have

$$\limsup_{n\to\infty} |f(z) - R_n(z)|^{\frac{1}{n+1}} \leq \limsup_{n\to\infty} (|I_1| + |I_2|)^{\frac{1}{n+1}}$$

$$\leq \max\left\{ \limsup_{n\to\infty} |I_1|^{1/(n+1)}, \limsup_{n\to\infty} |I_2|^{1/(n+1)} \right\}$$

$$= \max\left\{ r^s, \left(\frac{1}{R}\right)^{1-s} \right\}$$ ■

Corollary 3.10 — In the above conditions, the sequence $\{R_n(z)\}$ of interpolants converges to $f(z)$ uniformly and geometrically to $f(z)$ on $\mathbb{T}$.

A similar result can be established for the quasi-Hermite interpolants. Indeed, let $\{x_j\}_{j=1}^{n+1}$ be the zeros of $\chi_{n+1}(z) = \chi_{n+1}(z, w_n) = \varphi_{n+1}(z) + w_n \varphi_{n+1}^*(z)$, $(|w_{n+1}| = 1)$ and H_{2n} the quasi-Hermite interpolant in $\Lambda_{-n,n}$ satisfying

$$H_{2n}(x_j) = f(x_j), j = 1, \ldots, n+1$$

and

$$H'_{2n}(x_j) = f'(x_j), j = 1, \ldots, n+1, j \neq k, \quad \ldots (3.13)$$

k being any natural in $\{1, \ldots, n+1\}$. Then, it holds

Theorem 3.11 — *Under the conditions given above, one has*

$$\limsup_{n\to\infty} |f(z) - H_{2n}(z)|^{1/(n+1)} \leq \max\left(r, \frac{1}{R}\right) < 1$$

uniformly on $\mathbb{T}$. Thus, the sequence $\{H_{2n}\}$ converges to f(z) uniformly and geometrically on $\mathbb{T}$.

Starting from an analytic function in an annulus $\mathbb{B} = \{z : r < |z| < R, 0 < r < 1 < R\}$, we have proved uniform and geometric convergence of the interpolants on $\mathbb{T}$. Now, we might wonder if such set of convergence could be enlarged, i.e., if under the appropriate conditions, uniform and geometric convergence could take place in an annulus containing $\mathbb{T}$.. A positive answer will be given when certain symmetry conditions, both for the interpolants and for the annulus are satisfied.

Setting $\rho = \min\left(R, \frac{1}{r}\right)$ then the annulus $\mathbb{B}_\rho = \left\{z : \frac{1}{\rho} < |z| < \rho\right\} \subset \mathbb{B}$ and $f(z)$ is analytic in $\mathbb{B}_\rho$. Let $R_{2n} \in \Lambda_{-n,n}$ be the interpolant of $f(z)$ at the nodes $\{x_{j,2n}\}_{j=1}^{2n+1}$ such that $x_{j,2n}$ are the $(2n+1)$-roots of $w \in \mathbb{T}$. Then, it holds

Theorem 3.12 — *The sequence $\{R_{2n}\}$ converges to $f(z)$ uniformly and geometrically on compact subsets of* $\mathbb{B}_\rho = \left\{z : \frac{1}{\rho}, |z| < \rho\right\}$.

PROOF : Let K be a compact subset of $\mathbb{B}_\rho$, then one can write $K = K_1 \cup K_2$ where K_1 and K_2 can be empty. When both sets are nonempty the forthcoming analysis on K_2 reduces to K_1 making the change of variable $x = \frac{1}{z}$. Indeed, if $z \in K_2$ then $x \in K'_2 \subset \left\{x : \frac{1}{\rho} < |x| \leq 1\right\}$.

Set $\tilde{R}_{2n}(x) = R_{2n}\left(\frac{1}{x}\right) \in \Lambda_{-n,n}$ and $g(x) = f\left(\frac{1}{x}\right)$. It holds

$$\tilde{R}_{2n}(\tilde{x}_j) = \tilde{R}_{2n}\left(\frac{1}{x_j}\right) = R_{2n}(x_j) = f(x_j) = g\left(\frac{1}{x_j}\right) = g(\tilde{x}_j), j = 1, \ldots, 2n+1$$

Therefore,

$$\sup_{z \in K_2} |f(z) - R_{2n}(z)| = \sup_{x \in K'_2} |g(x) - \tilde{R}_{2n}(x)|.$$

$\tilde{R}_{2n}$ being the interpolant in $\Lambda_{-n,n}$ of $g(x)$ at the nodes $\bar{x}_j$, $j = 1, \ldots, 2n+1$ which are the $(2n+1)-st$ roots of $\bar{w}$. Observe that function g is clearly analytic in $\left\{ z : \frac{1}{\rho} < 1 \mid z \mid < \rho \right\}$.

Thus, take K_1 a compact such that $K_1 \subset \left\{ z : \frac{1}{\rho} < \mid z \mid \leq 1 \right\}$ which implies that there exist $r' > 0$ such that, for any $z \in K_1$:

$$\frac{1}{\rho} < r' \leq \mid z \mid \leq 1.$$

Now, we have, for all $z \in K_1$:

$$f(z) - R_{2n}(z) = \frac{1}{2\pi i} \frac{z^{2n+1} - w}{z^n} \left(\int_{C_{\frac{1}{\rho}}} \frac{t^n f(t)}{(t^{2n+1} - w)(t-z)} dt + \int_{C_\rho} \frac{t^n f(t)}{(t^{2n+1} - w)(t-z)} dt \right).$$

Set $M(f) = \sup_{t \in C_{\frac{1}{\rho}}} \{ |f(z)| \}$ then :

$$\mid f(z) - R_{2n}(z) \mid \leq \frac{2M(f)}{(r'^n} \left(\frac{1}{\rho^{n+1}} \frac{1}{1 - \frac{1}{\rho^{2n+1}}} \frac{1}{r' - \frac{1}{\rho}} + \frac{\rho^{n+1}}{(\rho^{2n+1} - 1)(\rho - 1)} \right)$$

$$= \frac{2M(f)}{\rho} \left(\frac{1}{\rho\left(r' - \frac{1}{\rho}\right)\left(1 - \frac{1}{\rho^{2n+1}}\right)} \left(\frac{1}{\rho r'}\right)^n + \left(\frac{1}{\rho r'}\right)^n \frac{1}{\left(1 - \frac{1}{\rho^{2n+1}}\right)(\rho - 1)} \right),$$

which yields, for all $z \in K_1$

$$\limsup_{n \to \infty} \mid f(z) - R_{2n}(z) \mid^{\frac{1}{2n+1}} \leq \frac{1}{\sqrt{\rho r'}}$$

Since $\rho r' > 1$, proof follows. ■

Let us next see that a similar result can be also deduced when considering more general distributions of interpolation nodes. More precisely, zeros of para-orthogonal polynomials will be used. For it, take into account that in the proof of the last Theorem interpolation on the nodes $\bar{x}_j, j = 1, \ldots, 2n+1$ was also required.

For this aim, assume that, in general, $x_1, \ldots, x_n$ are the zeros of the para-orthogonal polynomial $B_n(z, w) = B_n(z) = \rho_n(z) + w\rho_n^*(z)$, $w \in \Pi$, $\rho_n(z)$ being the nth monic Szegö polynomial with respect to a weight function $\omega(\theta)$ on $[0, 2\pi]$. Now the question is: are $\bar{x}_1, \ldots, \bar{z}_n$ zeros of a certain para-orthogonal polynomial?

If we assume that all the trigonometric moments $c_k = \int_0^{2\pi} e^{-ik\theta}\,\omega(\theta)\,d\theta$ are real then $\rho_n(z)$ has real coefficients. Therefore, if $Q_n(z) = (z - \bar{x}_1)\ldots(z - \bar{x}_n)$, then one can write

$$Q_n(z) = \lambda_n z^n B_n\left(\frac{1}{z}\right) = \lambda_n\,\omega\,(\rho_n(z) + \bar{w}\,\rho_n^*(z));\ \lambda_n \neq 0$$

Hence, $Q_n(z)$ is also para-orthogonal with respect to $\omega(\theta)$.

By Theorem 3.8 we can easily deduce the following -

Theorem 3.13 — *Let $\omega(\theta)$ be a weight function on $[0, 2\pi]$ with all its moments $c_k = \int_0^{2\pi} e^{-ik\theta}\,\omega(\theta\,d\theta$ real. Let $\{x_{j,2n}\}_{j=1}^{2n+1}$ be the zeros of $\chi_{n+1}(z) = \varphi_{n+1}(z) + w\varphi_{n+1}^*(z)$, $w \in \Pi$ and $\{\varphi_n(z)\}$ being the sequence of orthonormal polynomials for $\omega(\theta)$. Let $f(z)$ be analytic in $\left\{z : \frac{1}{\rho} < |z| < \rho, \rho > 1\right\}$ and let $R_{2n}(z)$ be the interpolant of $f(z)$ in $\Lambda_{-n,n}$ at the nodes $\{x_{j,2n}\}_{j=1}^{2n+1}$ Then, the sequence $\{R_{2n}(z)\}$ converges to $f(z)$ uniformly and geometrically on any compact K such that $k \subset \left\{z : \frac{1}{\rho} < |z| < \rho, \rho > 1\right\}$.*

Remark 3.14 : A family of weight functions with all its trigonometric moments real can be deduced as follows. Indeed, let $\nu(x)$ be a weight function on $[-1, 1]$ and set

$$\omega(\theta) = \nu(\cos\theta)\,|\sin\theta|,\ \theta \in [0, 2\pi]$$

so that $\nu(x)$ induces a weight function on $[0, 2\pi]$ which is symmetric in the sense that

$$z = e^{i\theta} : \int_0^{\pi} z^k\,\omega(\theta)\,d\theta + \int_0^{2\pi} \bar{z}^k\,\omega(\theta)\,d\theta;\ k \in \mathbb{Z}$$

Then, the trigonometric moments are real and given by

$$c_k = \int_0^{2\pi} e^{-ik\theta}\,\omega(\theta)\,d\theta = 2\int_0^{\pi} \cos k\theta\,\omega(\theta)\,d\theta$$

Remark 3.15 : The two last Theorems are also valid for quasi-Hermite interpolants as given in Theorem 3.11.

Remark 3.16 : If $f(z)$ is an entire function, then ρ can be taken as large as we wish. Therefore, convergence (both geometric and uniform) will occur in any compact $K \subset \mathbb{C} - \{0\}$.

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