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ON SEQUENCE-COVERING STRONG S-MAPPINGS

JINJIN LI[1] AND SHOULI JIANG[2]

[1]*Dept. of Math., Zhangzou Teachers' College, Fujian* 363000, *P.R. China*

[2]*Dept. of Math., Shang Dong University, Jinan* 250100, *P.R. China,*
E-mail : jiang@math.sdu.edu.cn

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In this paper, we give some new characterizations of the sequence-covering (resp. compact-covering or pseudo-sequence-covering), strong s-image of a locally separable metric space by means of cs-cover (resp. *k*-cover or cs^*-cover), and show that all these characterizations are mutually equivalent.

Key Words : Sequence-covering Mappings; Compact-covering Mappings; Strong s-mapping; CS-covers; K-covers

1. INTRODUCTION

E. Michael showed that a regular space has a countable *k*-network if and only if it is a compact-covering image of a separable metric space[1]. As a generalization of this theorem, S. Lin showed that a regular space has a locally countable *k*-network if and only if it is a compact-covering, strong s-image of a (locally separable) metric space[2]. Recently, Y. Tanaka and S. Xia obtained that a space has a locally countable *cs*-network if and only if it is a sequence-covering, strong *s*-image of a locally separable metric space[3]. In this paper, we give some new characterizations of the sequence-covering (resp. compact-covering or pseudo-sequence-covering), strong *s*-image of a locally separable metric space by means of *cs*-cover (resp. *k*-cover or cs^* cover), and show that all these characterizations are mutually equivalent.

First of all, we give some definitions

Definition 1 — Let $f: X \to Y$ be a mapping.

(1) f is called compact-covering[1] (resp. sequence-covering[4]) if each compact subset (resp. convergent sequence) of Y is an image of a compact subset (resp. convergent sequence) of X under f, where "convergent sequence" means the sequence with its limit while in [5], a mapping $f: X \to Y$ is called sequence-covering, if for each convergent sequence L in Y, there exists a compact subset (not necessarily a convergent sequence) K of X such that $f(K) = L$. Let us call such a sequence-covering mapping of [5] pseudo-sequence-covering in this paper.

(2) f is called a strong *s*-mapping[2] if for each $y \in Y$, there exists a neighbourhood V of y in Y such that $f^{-1}(V)$ is a separable subspace of X.

Definition 2 — Let $\mathcal{P}$ be a cover of a space X.

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(1) $\mathcal{P}$ is called a k-network for X[6] if, whenever $K \subset U$ with K compact and U open in X, then $K \subset \bigcup \mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$. If without requiring $\bigcup \mathcal{P}' \subset U$, then such a covering is called a k-cover of X. If each element of a k-cover $\mathcal{P}$ is closed in X, then $\mathcal{P}$ is called a closed k-cover of X. A space is an $\aleph_0$-space (resp. $\aleph$-space) if it has a countable (resp. σ-locally finite) k-network.

(2) $\mathcal{P}$ is called a cs-network[7] (resp. cs^*-network[8]) if whenever S is a sequence converging to a point $x \in X$, and U is a neighbourhood of x, there is $P \in \mathcal{P}$ such that $x \in P \subset U$, and P contains S eventually (resp. frequently). If without requiring $P \subset U$, such a covering is called a cs-cover (resp. cs^*-cover) of X.

In this paper, all spaces are regular and T_1, all mappings are continuous and onto.

2. Results

It is easy to show Lemmas 2.1 and 2.2.

Lemma 2.1 — Let $\mathcal{P}$ be a point-countable cs^*-cover of X. Then for each convergent sequence S, $S \subset U\mathcal{P}'$ for some finite $\mathcal{P}' \subset \mathcal{P}$.

Lemma 2.2 — Let $f: X \to Y$ and $g: Y \to Z$ be mappings.

(1) If f and g are compact-covering, then $g \circ f$ is compact-covering.

(2) If f is compact-covering and g is pseudo-sequence-covering, then $g \circ f$ is pseudo-sequence-covering.

Lemma 2.3 — The following are equivalent for a space X :

(1) X has a locally countable k-network.

(2) X has a locally countable cs-network.

(3) X has a locally countable cs^**-network*

Proof (1) $\Rightarrow$(2) — Let $\mathcal{R}$ be a locally countable k-network for X. By the regularity of X, we can assume that each element of $\mathcal{R}$ is closed in X. For each $x \in X$ there is a neighbourhood U_x of x in X such that U_x intersects at most countably many elements of $\mathcal{R}$. Put $\mathcal{P} = \{R \in \mathcal{R}: R \subset U_x$ for some $x \in X\}$. Then $\mathcal{P}$ is a star-countable k-network for X. Let $\{\mathcal{P}_\alpha : \alpha \in \Lambda\}$ be a decomposition of $\mathcal{P}$ (i.e., $\mathcal{P} = \bigcup \{\mathcal{P}_\alpha : \alpha \in \Lambda\}$, where each $\mathcal{P}_\alpha$ is a countable subcollection of $\mathcal{P}$ and for two distinct $\alpha, \beta \in \Lambda$, $(\bigcup \mathcal{P}_\alpha) \cap (\bigcup \mathcal{P}_\beta) = \phi$). For each $\alpha \in \Lambda$, let $\mathcal{J}_\alpha = \{\bigcup \mathcal{P}' :$ a finite $\mathcal{P}' \subset \mathcal{P}_\alpha\}$, $T_\alpha = \bigcup \mathcal{J}_\alpha$. Then $\{T_\alpha : \alpha \in \Lambda\}$ is a locally countable and disjoint cover of X. Let $\{x_n\}$ be a sequence on converging to x in X, there is a unique $\alpha \in \Lambda$ with $x \in T_\alpha$. Let U be a nbd of x in X, then $\{x\} \cup \{x_n : n \geq m\} \subset \bigcup \mathcal{P}' \subset U$ for some finite $\mathcal{P}' \subset \mathcal{P}$ and some $m \in N$. Let $\mathcal{P}'' = \{P \in \mathcal{P}' : x \in P\}$, then $\{x\} \cup \{x_n : n \geq i\} \subset X \bigcup (\mathcal{P}' \mathcal{P}'')$ for some $i \geq m$. Thus $\{x\} \cup \{x_n : n \geq i\} \subset \bigcup \mathcal{P}'' \subset U$. By $x \in \bigcap \mathcal{P}''$, we have that $\bigcup \mathcal{P}'' \in \mathcal{J}_\alpha$. Put $\mathcal{J} = \bigcup \{\mathcal{J}_\alpha : \alpha \in \Lambda\}$. This shows that $\mathcal{J}$ is a locally countable cs-network for X.

(2) $\Rightarrow$ (3) : By Definition 2.

(3) $\Rightarrow$ (1) : By Proposition 1.2 in [9] and Theorem 3.3 in [10].

Theorem 2.4 — *The following are equivalent for a space X :*

(1) X is a compact-covering, strong s-image of a locally separable metric space.

(2) X has a locally countable k-network.

(3) X has a locally countable closed k-cover consisting of $\aleph_0$-subspaces.

PROOF : (1) $\Leftrightarrow$ (2) — By Theorem in [2] or by Theorem 1.10 in [3]. (2) $\Rightarrow$ (3). Let $\mathcal{P}$ be a locally countable k-network for X. By the regularity of X, we can assume that each element of $\mathcal{P}$ is closed in X. For each $x \in X$, let V_x be a neighbourhood of x which meets only countably many elements of $\mathcal{P}$. Let $\mathcal{P}^* = \{P \in \mathcal{P}$ is contained in some $V_x\}$. Then $\mathcal{P}^*$ is a locally countable k-network for X, and each $P \in \mathcal{P}^*$ is an $\aleph_0$-subspace. Thus this $\mathcal{P}^*$ has the required property.

(3) $\Rightarrow$ (1) — Let $\mathcal{P}$ be a locally countable closed k-cover consisting of $\aleph_0$-subspaces of X, and put $\mathcal{P} = \{P_\alpha : \alpha \in \Lambda\}$. For each $\alpha \in \Lambda$, by Michael's Theorem in [1], there exists a separable metric space M_α such that $f_\alpha : M_\alpha \to P_\alpha$ is a compact-covering mapping. Let $M = \oplus \{M_\alpha : \alpha \in \Lambda\}$, $Y = \oplus \{P_\alpha : \alpha \in \Lambda\}$. Let $f = \oplus_{\alpha \in \Lambda} f_\alpha : M \to Y$ and $g : Y \to X$ be the obvious mapping, and $h = g \circ f : M \to X$ be the composition of f and g. Then M is a locally separable metric space. It will suffice to show that the following hold :

(a) f is compact-covering

For each compact subset K of Y, $K \subset \bigcup_{i=1}^{n} P_{\alpha_i}$ for some finitely many $\alpha_i \in \Lambda$. Since every P_{α_i} is both open and closed in Y, $K \cap P_{\alpha_i}$ is compact in P_{α_i}, and so $f_{\alpha_i}(L_i) = K \cap P_{\alpha_i}$ for some compact subset L_i of M_{α_i} for each $i \leq n$. Let $L = \oplus_{i+1}^{n} L_i$. Then L is compact in M with $f(L) = K$. Hence, f is compact-covering.

(b) g is compact-covering

For each compact subset K of X, since $\mathcal{P}$ is a closed k-cover of X, $K \subset \bigcup \mathcal{P}'$ for some finite $\mathcal{P}' \subset \mathcal{P}$. Let $L = \oplus \{P \cap K : P \in \mathcal{P}'\}$. Then L is compact in Y with $g(L) = K$, and so g is compact-covering.

(c) $h = g \circ f$ is a compact-covering, strong s-mapping.

By Lemma 2.2, h is compact-covering. So we must show that h is a strong s-mapping. For each $x \in X$, because $\mathcal{P}$ is a locally countable cover of X, there is a neighbourhood V_x of x in X such that $(\mathcal{P})_{V_x} = \{P \in \mathcal{P} : P \cap V_x \neq \phi\}$ is countable, and so $h^{-1}(V_x) \subset \oplus \{M_\alpha : \alpha \in \Lambda$ and $P_\alpha \cap V_x \neq \phi\}$ is a separable metric subspace of M. Therefore, h is a strong s-mapping.

Theorem 2.5 — *The following are equivalent for a space X :*

(1) X is a compact-covering, sequence-covering, strong s-image of a locally separable metric space.

(2) X is a sequence-covering, strong s-image of a locally separable metric space.

(3) X has a locally countable cs-network.

(4) X has a locally countable cs-cover consisting of $\aleph_0$-subspaces.

PROOF : (1) $\Rightarrow$ (2) : Obvious.

(2) $\Leftrightarrow$ (3) — By Theorem 1.7 in [3].

(3) $\Rightarrow$ (4) — As in the proof of theorem 2.4 (2) $\Rightarrow$ (3).

(4) $\Rightarrow$ (2) — Let $\mathcal{P}$ be a locally countable cs-cover consisting of $\aleph_0$-subspaces of X, and put $\mathcal{P} = \{P_\alpha : \alpha \in \Lambda\}$. For each $\alpha \in \Lambda$, by Michael's Theorem in [1], there exists a separable metric space M_α such that $f_\alpha : M_\alpha \to P_\alpha$ is a sequence-covering mapping. Let $M = \oplus \{M_\alpha : \alpha \in \Lambda\}$, $Y = \oplus \{P_\alpha : \alpha \in \Lambda\}$. Let $f = \oplus_{\alpha \in \Lambda} f_\alpha : M \to Y$ and $g : Y \to X$ be the obvious mapping, and $h = g \circ f : M \to X$ be the composition of f and g. Then M is a locally separable metric space. It will suffice to show that h is a sequence-covering, strong s-mapping.

(a) f is sequence-covering.

For any sequence $\{y_n\}$ with $y_n \to y$ in Y, $y \in P_\alpha$ for some $\alpha \in \Lambda$. Since P_α is open in Y, P_α contains $\{y_n\}$ eventually, so we can assume that $y_n \in P_\alpha$ for each $n \in N$. Then $y_n \to y$ in P_α. Because f_α is sequence-covering, there exists $x_n \in f_\alpha^{-1}(y_n)$ with $x_n \to x \in f_\alpha^{-1}(y)$ in M_α, and hence $x_n \in f^{-1}(y_n)$ with $x_n \to x$ in M. Thus f is sequence-covering.

(b) g is sequence-covering.

For any sequence $\{x_n\}$ with $x_n \to x$ in X, because $\mathcal{P}$ is a cs-cover of X, P_α contains $\{x_n\}$ eventually for some $\alpha \in \Lambda$. Pick $y_n \in g^{-1}(x_n) \cap P_\alpha$ when n is large enough. Then $y_n \to x \in g^{-1}(x) \cap P_\alpha$ in Y, and so g is sequence-covering.

(c) $h = g \circ f$ is a sequence-covering, strong s-mapping.

By proposition 2.2 in [4], h is sequence-covering. As in the proof of Theorem 2.4, it is easy to check that h is a strong s-mapping.

(3) $\Rightarrow$ (1) : By Lemma 2.3, X has a locally countable k-network. In view of Theorem 3.1 in [10], each compact subset of X is metrizable. From the proof of Lemma 1.3 in [11], (1) holds/

***Theorem* 2.6** — *The following are equivalent for a space X.*

(1) X is a pseudo-sequence-covering, strong s-image of a locally separable metric space.

(2) X has a locally countable cs-network.*

(3) X has a locally countable cs-cover consisting of $\aleph_0$-subspaces.*

PROOF : (1) $\Rightarrow$ (2) Suppose that $f : M \to X$ is a pseudo-sequence-covering, strong s-mapping, where M is a locally separable metric space. Then M has a star-countable base $\mathcal{B}$. It is easy to check that $f(\mathcal{B})$ is a locally countable cs^*-network for X.

(2) $\Rightarrow$ (3) — As in the proof of Theorem 2.4 (2) $\Rightarrow$ (3).

(3) $\Rightarrow$ (1) — Let $\mathcal{P}$ be a locally countable cs^*-cover consisting of $\aleph_0$-subspaces of X. We use the same notations as in the proof of Theorem 2.4 (3) $\Rightarrow$ (1). From the proof of Theorem 2.4 (3) $\Rightarrow$ (1), f is compact-covering. For each convergent sequence S, by Lemma 2.1, $S \subset \bigcup \mathcal{P}'$ for some finite $\mathcal{P}' \subset \mathcal{P}$. Let $K = \oplus \{P \cap S : P \in \mathcal{P}'\}$. Then K is sequentially compact in Y. Since Y is an $\aleph$-space, K is compact in Y by Chaber's Theorem in [12], and so $g(K) = S$. This shows g is pseudo-sequence-covering. By Lemma 2.2, h is pseudo-sequence-covering. As in the proof of Theorem 2.4, it is easy to check that h is a strong s-mapping.

Remark : (1) $\Leftrightarrow$ (2) shown in [13].

By Lemma 2.3 and Theorems 2.4 ~ 2.6, we have

Corollary 2.7 — The following conditions (a) ~ (c) are mutually equivalent for a space X

(a) Theorem 2.4 (1) ~ (3).

(b) Theorem 2.5 (1) ~ (4).

(c) Theorem 2.6 (1) ~ (3).

REFERENCES

1. E. Michael, *J. math. Mech.*, **15** (1966), 983-1002.
2. S. Lin, *Northeastern Math. J. (China)*, **4** (1988), 162-68.
3. Y. Tanaka and S. Xia, *Q & A in Gen. Topol.* **14** (1996), 217-31.
4. F. Siwiec, *General Topol. Appl.* **1** (1971), 143-54.
5. G. Gruenhage, E. Michael and Y. Tanaka, *Pacific J. Math.*, **113** (1984), 303-32.
6. P. O'Meara, *Proc. Amer. math. Soc.*, **29** (1971), 183-89.
7. J. A. Gutrie, *Gen. Topol. Appl.*, **1** (1971), 105-10.
8. Z. Gao, *Q & A in Gen. Topol.* **5** (1987), 271-79.
9. Y. Tanaka, *Topol. Proc.*, **123** (1987), 327-49.
10. D. Burke and E. Michael, *Pacific J. Math.*, **64** (1976), 79-92.
11. S. Lin and C. Liu, *Topol. Appl.* **74** (1996), 51-60.
12. J. Chaber, *Bull. Pol. Acad. Math.*, **24** (1976), 993-98.
13. S. Lin, *Generalized Metric Spaces and Mappings (in Chinese)*, Chinese Scientific Pub., Beijing, 1995.