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# EXISTENCE OF NONTRIVIAL SOLUTIONS TO NONLINEAR SYSTEMS OF HAMMERSTEIN INTEGRAL EQUATIONS AND APPLICATIONS

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The present paper employs the topological degree theory and the cone theory to investigate the existence of nontrivial solutions to nonlinear systems of Hammerstein integral equations and applies the obtained results to the two-point boundary value problems for systems of ordinary differential equations.

**Key Words : Systems of Hammerstein Integral Equations; Partial Ordering; Topological Degree; Nontrivial Solutions**

## 1. INTRODUCTION

In this paper, we shall investigate the system of Hammerstein integral equations :

$$\left.\begin{aligned}\varphi(x) &= \int_G k_1(x, y) f_1(y, \varphi(y), \psi(y))\, dy\\ \psi(x) &= \int_G k_2(x, y) f_2(y, \varphi(y), \psi(y))\, dy,\end{aligned}\right\} \quad \ldots (1.)$$

where $G$ is a bounded closed domain in the $N$-dimensional Eulidean space $R^N$, and its measure mes $G > 0$.

Due to the needs of solving practical problems, the research in systems of nonlinear integral equations is drawing more and more attention. However, rather strict conditions were imposed on the nonlinear terms in the previous results. For example, the following condition was used in [2] :

(H0) $f_i(x, u, v)$ $(i = 1, 2)$ are bounded blow and there exists continuous functions $a_i(x) > 0, b_i(x) \geq 0$ $(i = 1, 2)$ for $x \in G$, such that

$$f_1(x, u, v) \geq a_1(x)\, u - b_1(x), \;\; \forall x \in G, u \geq 0, v \in R^1$$

and

$$f_2(x, u, v) \geq a_2(x) v - b_2(x), \;\; \forall x \in G, v \geq 0, u \in R^1,$$

i.e. $f_i(x, u, v)$ $(i = 1, 2)$ are uniformly superlinear with respect to $u$ or $v$. But some simpler nonlinear terms such as $f_1(x, u, v) = u^2 + v^2, f_2(x, u, v) = u^2 - v^2$ can't satisfy the condition (H0).

In this paper, we established new existence theorems on the nontrivial solutions to the system (1), which can be used in dealing with the problems the literature[1&2] can't.

## 2. Main Results

For the sake of convenience, we first list some basic assumptions :

(H1) $k_1(x, y) \geq 0$ $(i = 1, 2)$, and there exists a constant $\tau > 0$ such that

$$\int_G k_1(x, y)\, dx \geq \tau \int_G k_2(x, y)\, dx; \quad \ldots (2)$$

(H2) $f_i(x, u, v)$ $(i = 1, 2)$ are bounded below, and there exist continuous functions $a(x) > 0$, $b(x) \geq 0$ and $d(x) \geq 0$ for $x \in G$ and a constant $\delta > 0$ such that

$$f_1(x, u, v) \geq a(x)u - b(x), x \in G, u, v \in R^1 \quad \ldots (3)$$

and

$$f_1(x, u, v) + d(x) \geq \delta f_2(x, u, v), x \in G, u, v \in R^1. \quad \ldots (4)$$

(H3) There exist continuous functions $c_i(x) \geq 0$ $(i = 1, 2)$ and a positive number $r_0 > 0$ such that

$$|f_i(x, u, v)| \leq c_i(x)(|u| + |v|), x \in G, |u| + |v| \leq r_0, \; i = 1, 2. \quad \ldots (5)$$

For any $\varphi, \psi \in C(G)$, let

$$K_1 \varphi(x) = \int_G k_1(x, y)\, a(y)\, \varphi(y) dy,$$

$$K_2 \psi(x) = \int_G k_2(x, y)\, \psi(y) dy,$$

then $K_1$ and $K_2$ are bounded linear operators on $C(G)$ with the spectral radii $r(K_1)$ and $r(K_2)$ respectively. If $r(K_i) \neq 0$ $(i = 1, 2)$, the famous Krein-Rutman's Theorem implies that there exist functions $g_i^*(x)$ $(i = 1, 2)$ such that $g_i^*(x) \geq 0$, $g_i^*(x) \not\equiv 0$ and

$$\int_G k_1(x, y)\, a(y)\, g_1^*(x)\, dx = r(K_1)\, g_1^*(y), \;\; \forall y \in G$$

and $$\int_G k_2(x, y)\, g_2^*(x)dx = r(K_2)\, g_2^*(y), \ \forall y \in G.$$

By $K_i$ satisfying $H$-Condition[3], we mean that there exists positive number $\delta_i > 0$ such that $K_i P_0 \subset P(g_i^*, \delta_i)$, where $P_0 = \{\varphi \in C(G) : \varphi(x) \geq 0, x \in G\}$, and

$$P(g_i^*, \delta_i) = \left\{ \varphi \in P_0 : \int_G g_i^*(x)\, \varphi(x)dx \geq \delta_i \| \varphi \| \right\} \ (i = 1, 2).$$

*Lemma* 1[3] — Linear operator $K_i$ satisfies the H-Condition if and only if there exists positive constant $\tau_i > 0$ such that $g^*(y) \geq \tau_i\, k(x, y)$, $\forall x, y \in G$.

Let

$$K_0\, \varphi(x) = \int_G [| k_1(x, y) | c_1(y) + | k_2(x, y, c_2(y)]\, \varphi(y)dy,$$

The spectral radius of $K_0$ is denoted by $r(K_0)$.

***Theorem*** 1 — *Let the assumptions (H1)-(H3) hold. If $K_1$ and $K_2$ satisfy the H-Condition. $r(K_1) > 1$ and $r(K_0) \leq 1$, then the system (1) has at least one nontrivial continuous solution.*

PROOF : Since $f_i(x, u, v)$ are bounded below, there exist $m_i \geq 0$ such that $f_i(x, u, v) + m_i \geq 0$ $(i = 1, 2)$. Let $D(x) = \int_G k_2(x, y)dy)\, dy$, $d_1(x) = m_1 \int_G k_1(x, y)\, dy$, and $d_2(x) = m_2 \int_G k_2(x, y)\, dy$. Let $P^* = P(g_1^*, \delta_1) \times P(g_2^*, \delta_2)$. It is obvious that $P^*$ is cone in $C(G) \times C(G)$. For any $(\varphi, \psi) \in C(G) \times C(G)$, it is clear that $A_i(\varphi, \psi)(x) + d_i(x) \geq 0$ for $i = 1, 2$. Since $K_i$ satisfy the H-Condition, so we have $A_i(\varphi, \psi) + d_i(x) \in P(g_i^*, \delta_i)$ $(i = 1, 2)$. Hence, $A(\varphi, \psi) + (d_1, d_2) \in P^*$ and it follows from (2) and (4) that,

$$\delta \int_G (A_2(\varphi, \psi)(x) + d_2(x))dx = \delta \int_G dx \int_G k_2(x, y) f_2(y, \varphi(y), \psi(y))\, dy + \delta \int_G d_2(x)\, dx$$

$$\leq \int_G [f_1(y, \varphi(y), \psi(y)) + d(y)]\, dy \int_G k_2(x, y)dx + \delta \int_G d_2(x)dx$$

$$\leq \int_G [f_1(y, \varphi(y), \psi(y)) + m_1]dy \int_G k_2(x, y)dx + \int_G D(x)dx + \delta \int_g d_2(x)dx$$

$$\leq \frac{1}{\tau} u \int_G k_1(x, y)\, [f_1(y, \varphi(y), \psi(y)) + m_1]\, dy + \int_G [D(x) + \delta d_2(x)]\, dx$$

$$= \frac{1}{\tau} \int_G [A_1(\varphi, \psi)(x) + d_1(x)]dx + \int_G [D(x) + \delta d_2(x)]\, dx.$$

Let $W = \int_G [D(x) + \delta d_2(x)]\,dx$, then

$$\int_G [A_1(\varphi, \psi)(x) + d_1(x)]\,dx \geq \tau\delta \int_G (A_2(\varphi, \psi)\,x) + d_2(x))\,dx - \tau W. \qquad \ldots (6)$$

Let $\varepsilon = r(K_1) - 1, M_2 = \| g_2^* \|$, and choose $R_0 > 0$ satisfying

$$R_0 > \max\left\{ \| D \| + \| d_1 \| + 2\delta \| d_2 \|, \| d_2 \| + \frac{WM_2}{\delta\delta_2} + \frac{W_2 \cdot mes\, G}{\varepsilon\tau\delta\delta_1\delta_2}\left[ \varepsilon\, \delta_1 \| d_1 \| + \varepsilon \int_G g_1^*(x)\, d_1(x) dx + x\, K_1) \int_G g_1^*(x) \frac{b(x)}{a(x)}\,dx \right]\right]$$

then for any $\forall(\varphi, \psi) \in C(G) \times C(G)$, $\| (\varphi, \psi) \| \geq R_0$ and $t \geq 0$, we have

$$(\varphi, \psi) - A(\varphi, \psi) \neq t\,(\varphi^* \cdot \theta), \qquad \ldots (7)$$

where $\varphi^*$ is a positive eigenvector of $K_1$ corresponding to $r(K_1)$. In fact, if there exist $(\varphi_0, \psi_0) \in C(G) \times C(G)$, $\| (\varphi_0, \psi_0) \| \geq R_0$, and $t_0 \geq 0$ such that $(\varphi_0, \psi_0) - A(\varphi_0, \psi_0) = t_0\,(\varphi^*, \theta)$, then

$$\varphi_0 - A_1(\varphi_0, \psi_0) = t_0\, \varphi^* \qquad \ldots (8)$$

and

$$\psi_0 - A_2(\varphi_0, \psi_0) = \theta. \qquad \ldots (9)$$

Hence, $\varphi_0 + d_1 \in P(g_1^*, \delta_1)$ and $\psi_0 + d_2 \in P(g_2^*, \delta_2)$, and (6), (8) and (9) yield

$$\int_G (\varphi_0(x) + d_1(x))\,dx \geq \tau\,\delta \int_G (\psi_0(x) + d_2(x))\,dx - \tau\, W.$$

Further, we have

$$\| \varphi_0 + d_1 \| \geq \frac{\tau\delta}{M_2 \cdot \text{mes}\, G} \int_G g_2^*(x)\,(\psi_0(x) + d_2(x))\,dx - \frac{\tau W}{\text{mes}\, G}$$

$$\geq \frac{\tau\delta\delta_2}{M_2 \cdot \text{mes}\, G} \| \psi_0 + d_2 \| - \frac{\tau W}{\text{mes}\, G}.$$

Therefore, $\| \varphi_0 + d_1 \| \geq \frac{\tau\delta\delta_2}{M_2 \cdot \text{mes}\, G} R_0 - \| d_1 \| - \frac{\tau\delta\delta_2}{M_2 \cdot \text{mes}\, G} \| d_2 \| - \frac{\tau W}{\text{mes}\, G}$ and then it follows from (3) that,

$$\int_G g_1^*(x)\,[\varphi_0(x) - A_1(\varphi_0, \psi_0)(x)]dx$$

$$= \int_G g_1^*(x)\,\varphi_0(x)dx - \int_G g_1^*(x)dx \int_G k_1(x,y) f_1(y, \varphi_0(y), \psi_0(y))dy$$

$$\leq \int_G g_1^*(x)\varphi_0(x)dx - \int_G g_1^*(x)dx \int_G k_1(x,y)a(y)\left[\varphi_0(y) - \frac{b(y)}{a(y)}\right]dy$$

$$= \int_G g_1^*(x)\,\varphi_0(x)dx - r(K_1) \int_G g_1^*(x)\left[\varphi_0(x) - \frac{b(x)}{a(x)}\right]dx$$

$$= -r(K_1) - 1) \int_G g_1^*(x)\,[\varphi_0(x) + d_1(x)]\,dx$$

$$+ (r(K_1) - 1) \int_G g_1^*(x)\,d_1(x)dx + r(K_1) \int_G g_1^*(x)\frac{b(x)}{a(x)}dx$$

$$\leq -\varepsilon\,\delta_1 \,\|\,\varphi_0 + d_1\,\| + \varepsilon \int_G g_1^*(x)\,d_1(x)\,dx + r(K_1) \int_G g_1^*(x)\frac{b(x)}{a(x)}dx$$

$$\leq -\frac{\varepsilon\tau\delta\delta_1\delta_2}{M_2\cdot \text{mes}\, G}R_0 + \varepsilon\,\delta_1 \,\|\,d_1\,\| + \frac{\varepsilon\tau\delta\delta_1\delta_2}{M_2 \cdot \text{mes}\, G}\,\|\,d_2\,\| + \frac{\varepsilon\tau\delta_1 W}{\text{mes}\, G}$$

$$+ \varepsilon \int_G g_1^*(x)d_1(x)dx + r(K_1) \int_G g_1^*(x)\frac{b(x)}{a(x)}dx < 0.$$

But, on the other hand, it follows from (8) that

$$\int_G g_1^*(x)\,[\varphi_0(x) - A_1(\varphi_0, \psi_0)(x)]\,dx = t_0 \int_G g_1^*(x)\,\varphi^*(x)dx \geq 0.$$

Now we arrive at a contrary. Hence (7) holds. By the lack of direction of topological degree, we obtain that $\forall R \geq R_0$,

$$\deg\,(I_A, B_R, \theta) = 0. \qquad \text{... (10)}$$

Use $B_{r_0}$ denoting $\{(\varphi, \psi) \in C(G) \times C(G) : \|\,\varphi\,\| + \|\,\psi\,\| < r_0\}$, then we assume, without loss of generality, that $A$ has no fixed point on $\partial B_{r0}$. Next we shall prove that for any $(\varphi, \psi) \in \partial B_{r0}$ and $\lambda \geq 1$ the following assertion holds:

$$A(\varphi, \psi) \neq \lambda(\varphi, \psi). \qquad \text{... (11)}$$

Otherwise, assume that there exist $(\varphi_0, \psi_0) \in \partial B_{r0}$ and $\lambda_0 \geq 1$ such that

$$A(\varphi_0, \psi_0) = \lambda_0\,(\varphi_0, \psi_0). \qquad \text{... (12)}$$

Obviously, $\lambda_0 > 1$. It follows from (5) and (12) that

$$\lambda_0 \mid \varphi_0(x) \mid \leq \int_G k_1(x, y)\, c_1(y)\, (\mid \varphi_0(y) \mid + \mid \psi_0(y) \mid dy$$

and

$$\lambda_0 \mid \varphi_0(x) \mid \leq \int_G k_2(x, y)\, c_2(y)\, (\mid \varphi_0(y) \mid + \mid \psi_0(y) \mid dy.$$

Hence,

$$\lambda_0 (\mid \varphi_0(x) \mid + \mid \psi_0(x) \mid) \leq \int_G (k_1(x, y)\, c_1(y) + k_2(x, y)\, c_2(y))\, (\mid \varphi_0(y) \mid + \mid \psi_0(y) \mid) dy.$$

Let $p(x) = \mid \varphi_0(x) \mid + \mid \psi_0(x) \mid$, then the above formula yields that $\lambda_0 p(x) \leq (K_0 p)(x)$, and then the Krein-Rutman's Theorem implies that $r(K_0) \geq \lambda_0 > 1$, contrary to $r(K_0) \leq 1$. Hence (11) holds. Consequently, we have

$$\deg\ (I - A, B_{r0}, \theta) = 1. \qquad \text{... (13)}$$

By virtue of the excision and solvability properties of the topological degree, (10) and (13) implies that $A$ has at least one fixed point in $B_{R_0} \setminus \overline{B_{r_0}}$, that is, the system (1) possesses nontrivial continuous solutions. □

In the following conclusions, we shall no more require $f_2(x, u, v)$ bounded below and $K_2$ satisfying the H-Condition. To this goal, we shall assume that

(H4) $k_1(x, y) \geq 0$, and there exists a positive constant $\tau > 0$ such that

$$k_1(x, y) \geq \tau \mid k_2(x, y) \mid, x, y \in G; \text{ and} \qquad \text{... (14)}$$

(H5) $f_1(x, u, v)$ is bounded below, and there exist continuous functions $a(x) > 0$, $b(x) \geq 0$ and $d(x, u) \geq 0$ and a constant $\delta > 0$ such that (3) holds and

$$f_1(x, u, v) + d(x, u) \geq \delta \mid f_2(x, y, v) \mid, x \in F, u, v \in R^1. \qquad \text{... (15)}$$

***Theorem*** 2 — *Let the assumptions (H3), (H4) and (H5) hold. If $K_1$ satisfies the H-Condition, $r(K_1) > 1$ and $r(K_0) < 1$, then the system (1) possesses at least one nontrivial continuous solution.*

PROOF : Since $f_1(x, u, v)$ is bounded below, there exists a constant $m_1 \geq 0$ such that $f_1(x, u, v) + m_1 \geq 0$. Let $d_1(x) = m_1 \int_G k_1(x, y) dy$, then for any $(\varphi, \psi) \in C(G) \times C(G)$, we have $A_1(\varphi, \psi)(x) + d_1(x) \geq 0$. Since $K_1$ satisfies the H-Condition, we get $A_1(\varphi, \psi) + d_1(x) \in P(g_1^*, \delta_1)$. For any $\varphi \in C(G)$, let

$$D\, \varphi(x) = \int_G k_1(x, y)\, d(y, \varphi)(y))\, dy,$$

then $D: C(G) \to P_0$ is completely continuous, and then for $(\varphi, \psi) \in C(G) \times C(G)$, it follows from (14) and (15) that (obviously, $f_1(x, u, v) + d(x, u) \geq 0$, $x \in G$, $u, v \in R^1$)

$$\tau \delta | A_2(\varphi, \psi)(x) | \leq \tau \delta \int_G | k_2(x, y) f_2(y, \varphi(y), \psi(y)) | \, dy$$

$$\leq \tau \int_G | k_2(x, y) | [f_1(y, \varphi(y), \psi(y)) + d(y \, \varphi(y))] \, dy$$

$$\leq \int_G k_1(x, y) [f_1(y, \varphi(y), \psi(y)) + d(y, \varphi(y))] \, dy$$

$$= A_1(\varphi, \psi)(x) + D \varphi(x).$$

Set $\varepsilon = r(K_1) - 1$ and choose $R_0 > 0$ satisfying

$$R_0 > \frac{1}{\delta_1} \int_G g_1^*(x) \, d_1(x) \, dx + \frac{1+\varepsilon}{\varepsilon \delta_1} \int_G g_1^*(x) \frac{b(x)}{a(x)} \, dx.$$

It is not difficult to show that $R_1 > \max\{\| d_1 \|, R_0\}$ can be chosen sufficiently large such that $\| \varphi + d_1 \| \geq R_0$ for any $(\varphi, \psi) \in C(G) \times C(G)$ satisfying $\| (\varphi, \psi) \| \geq R_1$. $\varphi + d_1 \geq 0$, and $\tau \delta | \psi | \leq \varphi + D \varphi$.

We shall next prove that (7) holds for $\forall (\varphi, \psi) \in C(G) \times C(G)$, $\| (\varphi, \psi) \| \geq R_1$, and $t \geq 0$. In fact, if not, there exist $(\varphi_0, \psi_0) \in C(G) \times C(G)$, $\| \varphi_0, \psi_0) \| \geq R_1$, and $t_0 \geq 0$ such that $(\varphi_0, \psi_0) - A(\varphi_0, \psi_0) = t_0 (\varphi^*, \theta)$, and then

$$\varphi_0 - A_1(\varphi_0, \psi_0) = t_0 \varphi^*$$

and

$$\psi_0 - A_2(\varphi_0, \psi_0) = \theta.$$

Hence $\varphi_0 + d_1 \in P(g_1^*, \delta_1)$, $\psi_0 = A_2(\varphi_0, \psi_0)$, and (16) yields

$$\tau \delta | \psi_0(x) | = \tau \delta | A_2(\varphi_0, \psi_0)(x) | \leq A_1(\varphi_0, \psi_0)(x) + D \varphi_0(x) \leq \varphi_0(x) + D \varphi_0(x).$$

Therefore, $\| \varphi_0 \| \geq R_0$, and by $K_1$ satisfying the H-Condition, it follows from (3) that

$$\int_G g_1^*(x) [\varphi_0(x) - A_1(\varphi_0, \psi_0)(x)] \, dx$$

$$= \int_G g_1^*(x) \, \varphi_0(x) dx - \int_G g_1^*(x) dx \int_G k_1(x, y) f_1(y, \varphi_0(y), \psi_0(y)) \, dy$$

$$\leq \int_G g_1^*(x)\,\varphi_0(x)dx - \int_G g_1^*(x)\,dx \int_G k_1(x,y)\,a(y)\left[\varphi_0(y) - \frac{b(y)}{a(y)}\right]dy$$

$$= \int_G g_1^*(x)\,\varphi_0(x)dx - r(K_1)\int_G g_1^*(x)\left[\varphi_0(x) - \frac{b(x)}{a(x)}\right]dx$$

$$= -\varepsilon \int_G g_1^*(x)\,[\varphi_0(x) + d_1(x)]\,dx$$

$$+\ \varepsilon \int_G g_1^*(x)d_1(x)dx + (1+\varepsilon)\int_G g_1^*(x)\frac{b(x)}{a(x)}dx$$

$$\leq -\varepsilon\,\delta_1 \,\|\, \varphi_0 + d_1 \,\|\, + \varepsilon \int_G g_1^*(x)d_1(x)dx + (1+\varepsilon)\int_G g_1^*(x)\frac{b(x)}{a(x)}dx$$

$$< 0.$$

The rest of the proof is similar to that in theorem 1, so we omit it. □

## 3. APPLICATIONS

Consider the existence of nontrivial solutions to the following system of second order ordinary differential equations

$$\begin{Bmatrix} -\varphi''(x) = \varphi^2(x) + \psi^2(x) \\ -\varphi''(x) = 2\varphi^2(x) - \psi^2(x) \end{Bmatrix} \qquad \text{... (17)}$$

associated with the boundary conditions

$$\varphi(0) = \varphi'(1) = 0,\ \psi(0) = \psi(1) = 0. \qquad \text{... (18)}$$

It is well known that the boundary value problem (17)-(18) has nontrivial solutions in $C^2[0, 1]$ if and only if the following system of integral equations has nontrivial solutions in $C[0, 1]$:

$$\left.\begin{aligned} \varphi(x) &= \int_0^1 k_1(x,y) f_1(y, \varphi(y), \psi(y))\,dy \\ \text{and} \qquad \psi(x) &= \int_0^1 k_2(x,y) f_2(y, \varphi(y), \psi(y))\,dy, \end{aligned}\right] \qquad \text{... (19)}$$

where

$$k_1(x,y) = \begin{cases} x, & x \leq y \\ y, & y < x \end{cases} \quad k_2(x,y) = \begin{cases} x(1-y), & x \leq y \\ y(1-x), & y < x. \end{cases}$$

$$f_1(x, u, v) = u^2 + v^2,\ f_2(x, u, v) = 2u^2 - v^2.$$

Clearly, $0 \le k_2(x, y) \le k_1(x, y)$, $|f_2(x, u, v)| \le f_1(x, u, v) + u^2$, and $f_1(x, u, v) \ge u - 1$, for $x, y \in [0, 1]$ and $u, v \in R^1$. Hence, the assumptions (H4) and (H5) hold. In addition, it is easy to see that for arbitrarily small positive number $c > 0$, there exists a $r_0 > 0$ such that $|f_i(x, u, v)| \le c(|u| + |v|)$, $x \in [0, 1]$, $i = 1, 2$, whenever $|u| + |v| \le r_0$. So the assumption (H3) holds.

Let $K_1\varphi(x) = \int_0^1 k_1(x, y)\varphi(y)dy$, then the theory of ordinary differential equations implies that $r(K_1) = \frac{\pi^2}{4} > 1$, $g_1(x) = \sin\left(\frac{\pi x}{2}\right)$ and

$$\int_0^1 k_1(x, y)\, g_1(x)\, dx = r(K_1)\, g_1(y).$$

Let $\alpha_1(x) = x$, then it is easy to verify that $\forall x, y, z \in [0, 1]$, $k_i(x, y) \ge \alpha_1(x)\, k_1(z, y)$. In virtue of this fact, if we use $\beta_1$ denoting $r(K_1)\int_0^1 \alpha_1(x) g_1(x) dx$, then $g_1(x) \ge \beta_1 k_1(z, x)$ for any $x, z \in [0, 1]$ and $K_1 : P_0 \to P(g_1^*, \delta_1)$, i.e. $K_1$ satisfies the H-Condition, where $\delta_1 = r(K_1)\beta_1$.

By above arguments, Theorem 2 immediately implies the following :-

***Theorem* 3** — *The boundary value problem (17) - (18) has at least one nontrivial solution in* $C^2[0, 1]$.

*Remark* 1 : Our conclusions are essentially different from those in [1, 2]. The literature [1, 2] cannot deal with such nonlinear terms as in (17).

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