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QUADRATURES FOR CAUCHY PRINCIPAL VALUE INTEGRALS USING CUBIC SPLINE INTERPOLATIONS

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Quadrature formulas based on product integrations have been obtained for the numerical evaluation of Cauchy principal value integrals using cubic spline interpolations on arbitrary spaced knots. The present quadrature formulas are suitable for arbitrary weight functions and when the function is not given explicitly but only by data points on arbitrary spaced knots. Computational procedures of the quadrature formulas develop interest.

Key Words : Cauchy Principal Value Integrals; Cubic Spline Interpolation; Product Integration

1. INTRODUCTION

Let $I(gf; \tau)$ denote the integral in the Cauchy principal value sense of the functions f and g, defined by

$$I(gf, \tau) = \int_a^b g(x)(f(x)/(x-\tau))\, dx, \ a < \tau < b, \qquad \dots (1)$$

where g may be an admissible weight function on (a, b) or an arbitrary integrable function subject to certain conditions such that the integral (1) exists. An interpolatory product rule for the evaluation of (1) has been discussed in Rabinowitz[8].

Rules for the numerical evaluation of (1) and its convergence, using cubic spline interpolations have been discussed in Dagnino & Santi[3 & 4], for the case when $g(x)$ is non-negative weight function on (a, b) such that $\fint_a^b (g(x)/(x-\tau))\, dx$ exists. Convergence of the present quadrature formulas for a suitable class of functions can be established by proceeding similar to Dagnino & Santi[4] and Rabinowitz[8].

2. Spline Interpolations

If the points $\{x_i\}$ are given on a mesh defined by $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ not necessarily to be equally spaced, then the cubic spline interpolations $f_k(x)$ and $g_k(x)$ for $f(x)$ and $g(x)$ on the interval $x_k \le x \le x_{k+1}$, $k = 0(1)\, n-1$ may be written as:-

$$f_k(x) = \sum_{r=0}^{3} a_{k,r} (x - x_k)^r, \quad k = 0\ (1)\ n-1, \qquad \ldots (2)$$

and

$$g_k(x) = \sum_{s=0}^{3} b_{k,s} (x - x_k)^s, \quad k = 0\ (1)\ n-1. \qquad \ldots (3)$$

Natural splines are frequently used, as they have minimum total curvature among all sufficiently smooth interpolating curves. For natural splines, we give below the determining equations for $a_{k,r}$ and $b_{k,s}$ (see Davis & Rabinowitz [5, pp. 55]) :

Determining equations for $a_{k,r}$

$$a_{k,1} = f(x_k), \quad k = 0(1)\ n,$$

$$a_{k,2} = \{(a_{k+1,1} - a_{k,1})/h_k\} - \{2a_{k,3} + a_{k+1,3})\, h_k/3\}, \quad k = 0\ (1)\ n-1,$$

$$a_{k,4} = (a_{k+1,3} - a_{k,3})/(3h_k), \quad k = 0(1)\ n-1$$

and

$$h_{k-1}\, a_{k-1,3} + 2(h_{k-1} + h_k)\, a_{k,3} + h_k a_{k+1,3} = 3[\{(a_{k+1,1} - a_{k,1})/h_k\} - \{(a_{k,1} - a_{k-1,1})/h_{k-1}\}], \quad k = 1\ (1)\ n-1;\ a_{0,3} = a_{n,3} = 0.$$

Determining Equations for $b_{k,s}$

$$b_{k,1} = g(x_k), \quad k = 0(1)n,$$

$$b_{k,2} = \{(b_{k+1,1} - b_{k,1})/h_k\} - \{2b_{k,3} + b_{k+1,3})\, h_k/3\}, \quad k = 0\ (1)\ n-1,$$

$$b_{k,4} = (b_{k+1,3} - b_{k,3})/(3h_k), \quad k = 0\ (1)\ n-1,$$

$$h_{k-1}\, b_{k-1,3} + 2(h_{k-1} + h_k)\, b_{k,3} + h_k\, b_{k+1,3} = 3[\{(b_{k+1,1} - b_{k,1})/h_k\} - \{(b_{k,1} - b_{k-1,1})/h_{k-1}\}]$$

and

$$k = 1\ (1)\ n-1;\ b_{0,3} = b_{n,3} = 0.$$

For other specific conditions see Gerald and Wheatley [6, pp. 233-240]. For calculations of cubic smoothing splines for equally spaced data, see Cuplin[2].

3. Spline Product Quadrature Rules

We derive our quadrature formula by approximating the integral (1) in the following discrete form

$$I(gf;\tau)=\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} g(x)\,(f(x)/(x-\tau))\,dx \approx S_n(gf;\tau)=\sum_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} g(x)\,(f_k(x)/(x-\tau))\,dx,\ a<\tau<b. \quad \dots (4)$$

Using (2) in (4), we obtain

$$I(gf;\tau)\approx S_n(gf;\tau)=\sum_{k=0}^{n-1}\sum_{r=0}^{3} a_{k,r}A_{k,r}(g;\tau), \quad \dots (5)$$

where

$$A_{k,r}(g;\tau)=\int_{x_k}^{x_{k+1}} g(x)\,((x-x_k)^r/(x-\tau))\,dx,\ r=0(1)3. \quad \dots (6)$$

For the evaluations of (1) two quadrature rules based on the two ways computations of $A_{k,r}(g;\tau)$ are given below :

Quadrature Rule I — In (6) replacing $g(x)$ by (3), we obtain

$$A_{k,r}(g;\tau)\approx Q_{k,r}(g,\tau)=\sum_{s=0}^{3} b_{k,s}B_{k,r+s}(\tau),\ r=0(1)3,\ k=0(1)\ n-1, \quad \dots (7)$$

where

$$B_{k,r+s}(\tau)=⨍_{x_k}^{x_{k+1}} (x-x_k)^{r+s}/(x-\tau))\,dx, \quad \dots (8)$$

$r+s$ is an integer such that $0\le r+s\le 6$. $B_{k,m}(\tau)$ for $m = 1\ (1)\ 6$ can be computed from the following relation:

$$B_{k,m}(\tau)=((x_{k+1}-x_k)^m/m)+(\tau-x_k)\,B_{k,m-1}(\tau),\ m=1\ (1)\ 6,\ k=0\ (1)\ n-1, \quad \dots (9)$$

where

$$B_{k,0}(\tau)=\log|(x_{k+1}-\tau)/(x_k-\tau)|,\ k=0\ (1)\ n-1.$$

Quadrature Rule II — $A_{k,r}(g;\tau)$ for $r = 1\ (1)\ 3$ can be computed in terms of $A_{k,0}(g;\tau)$ from the following relation

$$A_{k,r}(g;\tau)=C_{k,r-1}(g;\tau)+(\tau-x_k)A_{k,r-1}(g;\tau),\ k=0\ (1)\ n-1, \quad \dots (10)$$

where

$$C_{k,m}(g;\tau)=\int_{x_k}^{x_{k+1}} g(x)\,(x-x_k)^m\,dx,\ m=0\ (1)\ 2,\ k=0\ (1)\ n-1. \quad \dots (11)$$

$A_{k,0}(g;\tau), k=0(1)\,n-1$, are non-singular integrals for $\tau\notin(x_k,x_{k+1})$. Out of n intervals $(x_k,x_{k+1}), k=0(1)n-1, \tau\in(x_k,x_{k+1})$ for only one value of k. $A_{k,0}(g;\tau)$ for $\tau\in(x_k,x_{k+1})$ can be computed to desired degree of accuracy using quadrature formulas for Cauchy principal value integrals described in Chawla & Kumar[1] and Kumar[7]. $A_{k,0}(g;\tau)$, for $\tau\notin(x_k,x_{k+1})$ and $C_{k,m}(g;\tau), m=0\ (1)\ 2,\ k=0\ (1)\ n-1$ are non-singular integrals, therefore they can be computed to desired degree of accuracy using suitable quadrature formulas, already existing in the literature.

To illustrate the above quadrature formulas, we evaluate the integral

$$I=\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}(e^x/(x-0.5))\,dx$$

for $\alpha=\beta=-3/4$ and $\alpha=-\beta=1/2$, by quadrature rule I and II; the corresponding approximate values of I obtained for n = 4(2) 8, using equispaced abscissas are shown in Table I.

TABLE I

N	Quadrature Rule I		Quadrature Rule II	
	$\alpha=\beta=-3/4$	$\alpha=-\beta=1/2$	$\alpha=\beta=-3/4$	$\alpha=-\beta=1/2$
4	10.238561	−0.1872562	10.2385163576	−0.1827361295
6	10.239834	−0.1829382	10.2391692739	−0.1830729539
8	10.239115	−0.1830524	10.2391636179	−0.1830791895
Exact I	10.239164	−0.1830792	10.2391636107	−0.1830791871

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