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WEIGHTED COMPOSITION OPERATORS ON WEIGHTED SPACES IN THE NON-LOCALLY CONVEX SETTING

BHOPINDER SINGH* AND KAMALJEET KOAUR**

**Department of Mathematics, Govt. College of Engineering and Technology, Old University Campus, Canal Road, Jammu* 180 001, *India*

***Department of Mathematics, Govt. M.A.M. College, Jammu* 180 004, *India*

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For a system V of weights on a topological space X and a Hausdorff topological algebra E, let $FV_b(X, E)$ and $FV_0(X, E)$ be the weighted spaces of E-valued functions f on X for which $vf(X)$ is bounded in E and vf vanishes at infinity on X, respectively. Let $\pi : X \to E$ and $T : X \to X$ be two mappings. Then we give necessary and sufficient conditions for the pair (π, T) to induce weighted composition operators on $FV_b(X, E)$ and $FV_0(X, E)$. These results extend the recent work of Khan and Thaheem[7] to a general setting of vector-valued functions and of Singh and one of the authors[15] to a non-locally convex setting.

Key Words : Weighted Composition Operators; Weighted Topology, System of Weights; Topological Vector Space; Locally Idempotent Algebra; Shrinkable Neighbourhoods

1. INTRODUCTION

Weighted composition operators on weighted locally convex spaces of vector-valued continuous functions have been studied by Singh, Summers, Manhas and one of the authors[12, 14, 15 & 17]. For details, we refer to the recent survey article of Singh and one of the authors[16], as well as the monograph[13] by Singh and Manhas. Recently, Khan and Thaheem[7] generalizes the work of Singh and Manhas[12] to weighted spaces of continuous functions in the non-locally convex framework. The purpose of this paper is to extend the results of Singh and Singh[15], and Khan and Thaheem[7] to the weighted spaces $FV_b(X, E)$ and $FV_0(X, E)$ of E-valued functions on X which are not necessarily continuous, and where X is just a topological space and the range space E is a topological vector space (or a topological algebra) which is not necessarily locally convex.

2. PRELIMINARIES

Throughout this paper we shall assume, unless stated otherwise, that X is a Hausdorff topological space, and E is a non-trivial Hausdorff topological vector space. Let $F(X, E)$ be the vector space of all E-valued functions on X, and let $F_b(X, E)$ $(F_0(X, E))$ denote the subspace of $F(X, E)$ consisting of those functions which are bounded (vanish at infinity). Let $S^+(X)$ denote the set of all non-negative upper semi-continuous functions on X, and let $S_0^+(X)$ $(S_c^+(X))$ be the subset of $S^+(X)$ consisting of those functions which vanish at infinity (have compact support). A system of weights on X is a subset V of $S_0^+(X)$ such that V is directed upward (or a Nachbin family) and $V > 0$ (i.e., for given $u, v \in V$ and $\alpha > 0$, there exists $w \in V$ such that $\alpha u, \alpha v \le w$ (pointwise on X); and for every $x \in X$ there is $v_x \in V$ such that $v_x(x) > 0$).

Given such a V, let $FV_b(X, E)$ $(FV_0(X, E))$ denote the subspace of $F(X, E)$ consisting of those f such that $vf \in F_b(X, E)$ $(F_0(X, E))$ for all $v \in V$. On $FV_b(X, E)$, we shall consider the weighted topology w_V of 0 consisting of all sets of the form :

$$S(v, G) = \{f \in FV_b(X, E) : v(x)f(x) \in G \text{ for all } x \in X\}$$

where $v \in V$ and G is a neighbourhood of 0 in E.

The space $FV_b(X, E)$ endowed with w_V is called a weighted space of vector-valued functions. The space $FV_0(X, E)$, being a subspace of $FV_b(X, E)$, is equipped with the topology induced by $FV_b(X, E)$.

The following are some instances of weighted spaces :

(1) If $V = K^+(X)$, the set of all non-negative constant functions on X, then $FV_b(X, E) = F_b(X, E)$ and $FV_0(X, E) = F_0(X, E)$. The topology w_V in this case is the topology σ of uniform convergence.

(2) If $V = \{\alpha \chi_k : \alpha \ge 0 \text{ and } K \subset X, K \text{ compact}\}$ and $V = S_c^+(X)$, then $FV_b(X, E) = FV_0(X, E) = F(X, E)$ and w_V is the compact-open topology k.

(3) If $V = S_0^+(X)$, then $FV_b(X, E) = FV_0(X, E) = F_b(X, E)$ and w_V in this case is called the substrict topology β_0.

For more details on weighted spaces, we refer to Summers[18], Prolla[11], Bierstedt[1] and Bierstedt, Meise and Summers[2] when E is a scalar field or a locally convex space, and to Bierstedt[1] Khan[3-6] and Nawrocki[10] in the general setting.

Let $\theta : X \to \mathbb{C}$ and $\pi : X \to E$ be two mappings. Then the scalar multiplication on E and, in case E is an algebra, multiplication on E give rise to two linear transformations M_θ and M_π from $FV_b(X, E)$ $(FV_0(X, E))$ into $F(X, E)$ defined by $M_\theta(f) = \theta f$ and $M_\pi(f) = \pi f$, where the product of functions is defined pointwise. In case M_θ and M_π takes $FV_b(X, E)$ $(FV_0(X, E))$ into itself and are continuous, they are called multiplication operator on $FV_b(X, E)$ $(FV_0(X, E))$ induced by θ and π, respectively. Let T be a selfmap on X. Then the correspondence $f \to f \circ T$ is a linear transformation from $FV_b(X, E)$ $(FV_0(X, E))$ into $F(X, E)$ and is denoted by C_T. In case C_T takes $FV_b(X, E)$ $(FV_0(X, E))$ into itself and is also continuous, it is called the composition operator on $FV_b(X, E)$ $(FV_0(X, E))$ induced by T. The product $M_\pi C_T$ of multiplication operator M_π on $FV_b(X, E)$ $(FV_0(X, E))$ and composition operator C_T on $FV_b(X, E)$ $(FV_0(X, E))$ is called the weighted composition operator on $FV_b(X, E)$ $(FV_0(X, E))$ induced by the pair (π, T) and we denote it by $W_{\pi, T}$.

A neighbourhood G of 0 in E is called shrinkable if $r\bar{G} \subseteq \text{int}.G$ for $0 \le r < 1$. It is known from Klee [8, Theorems 4 and 5] that every Hausdorff topological vector space has a base of shrinkable neighbourhoods of 0 and also the Minkowski functional p_G of any such neighbourhood

G is continuous. A topological algebra E is called locally idempotent algebra if it has a base $\mathcal{N}$ of neighbourhoods of 0 consisting of idempotent sets (G is called idempotent or multiplicative if $G^2 = G \cdot G \subseteq G$). It is easily seen that if $G \in \mathcal{N}$ is idempotent, then p_G is submultiplicative, that is, $p_G(st) \leq p_G(s)\, p_G(t)$ for all $s, t \in E$. The notion of locally idempotent algebras is a strict generalization of the notion of locally multiplicatively convex algebras introduced by Michael[9] (see also Zelazko[19]).

3. Weighted Composition Operators

In this section, we present necessary and sufficient conditions for $W_{\pi, T}$ and $W_{\theta, T}$ to be the weighted composition operators on weighted spaces $FV_b(X, E)$ and $FV_0(X, E)$. We begin with the following proposition :

***Proposition* 3.1** — *Let E be a Hausdorff topological algebra, and let $\mathcal{N}$ be a base of neighbourhoods of 0 in E. If $\pi : X \to E$ and $T : X \to X$ are two mappings such that $W_{\pi, T} : FV_0(X, E) \to FV_b(X, E)$ $FV_b(X, E)$ is a weighted composition operator, then for every $v \in V$ and $G \in \mathcal{N}$, there exist $u \in V$ and $H \in \mathcal{N}$ such that*

$$v(x)\, P_G(\pi(x)t) < u_0\, T(x)\, p_H(t) \text{ for all } x \in X \text{ and } t \in E.$$

PROOF : We may assume that $\mathcal{N}$ consists of closed, balanced, and shrinkable sets.

Let $v \in V$ and $G \in \mathcal{N}$. Then, by continuity of $W_{\pi, T}$, there exist $u \in V$ and $H \in \mathcal{N}$ such that

$$W_{\pi, T}(S(u, H) \cap FV_0(X, E)) \subseteq S(v, G). \qquad \text{... (1)}$$

We claim that $v(x)\, p_G(\pi(x), t) \leq 3u \circ T(x)\, p_H(t)$ for all $x \in X$ and $t \in E$. Assume that for some $x_0 \in X$ and $t_0 \in E$,

$$v(x_0)\, p_G(\pi(x_0)\, t_0) > 3u \circ T(x_0)\, p_H(t_0).$$

Let $h : X \to E$ be defined as $h(x) = 3t_0$ for $x = T(x_0)$, and $= 0$ otherwise. Then $h \in FV_0(X, E)$. Let $\alpha = v(x_0)\, p_G(\pi(x_0)\, t_0)$ and define $f = h/\alpha$. Since p_H is homogeneous, for any $x \in X$,

$$p_H(u(x) f(x)) \leq 3/\alpha\, u \circ T(x_0)\, p_H(t_0) < 1,$$

or equivalently, $u(x) f(x) \in H$. So we have $f \in S(u, H)$. Hence, by (1), $W_{\pi, T} f \in S(v, G)$. This implies that, for any $x \in X$,

$$p_G(v(x)\, \pi(x) f \circ T(x)) \leq 1.$$

In particular, by taking $x = x_0$, we have

$$v(x_0)\, p_G(\pi(x_0)\, t_0) \leq \alpha/3 = 1/3\; v(x_0)\, p_G(\pi(x_0)\, t_0),$$

which is absurd. This proves our claim and the proposition.

Theorem 3.2 — *Let $\mathcal{N}$ be a base of neighbourhoods of 0 in a Hausdorff topological algebra E. Then for mappings $\pi : X \to E$ and $T : X \to X$, the following are equivalent :*

(a) $W_{\pi, T}$ is a weighted composition operator on $FV_b(X, E)$; and

(b) for every $v \in V$ and $G \in \mathcal{N}$ there exist $u \in V$ and $H \in \mathcal{N}$ such that

$$v(x)\, p_G(\pi(x)(t) < u \circ T(x)\, p_H(t) \text{ for all } x \in X \text{ and } t \in E.$$

PROOF : We may assume that $\mathcal{N}$ consists of closed, balanced and shrinkable sets.

(a) $\Rightarrow$ (b) — This implication follows from Proposition 3.1 above.

(b) $\Rightarrow$ (a) — We first show that $W_{\pi, T}$ takes $FV_b(X, E)$ into itself.

Let $f \in FV_b(X, E)$, and let $v \in V$ and $G \in \mathcal{N}$. Choose $u \in V$ and $H \in \mathcal{N}$ such that

$$v(x)\, p_G(\pi(x)t) < u \circ T(x)\, p_H(t) \qquad \text{... (1)}$$

for all $x \in X$ and $t \in E$. There exists a scalar $k > 0$ such that $(uf)(X) \subseteq kH$, that is, $p_H(u(x) f(x)) \leq k$ for all $x \in X$. Now

$$p_G(v(x)\, \pi(x) f \circ T(x)) \leq u \circ T(x)\, p_H(f \circ T(x)) \leq k$$

for all $x \in X$ implies that $W_{\pi, T} f \in FV_b(X, E)$.

Next, we establish the continuity of $W_{\pi, T}$. For this, let $\{f_\alpha : \alpha \in I\}$ be a net in $FV_b(X, E)$ such that $f_\alpha \to 0$. Also let $v \in V$ and $G \in \mathcal{N}$, and choose $u \in V$ and $H \in \mathcal{N}$ as above satisfying (1). There exists an $\alpha_0 \in I$ such that $f_\alpha \in S(u, H)$ for all $\alpha \geq \alpha_0$. Now, for any $x \in X$ and $\alpha \geq \alpha_0$, we have

$$p_G(v(x)\, \pi(x) f_\alpha \circ T(x)) \leq u \circ T(x)\, p_H(f_\alpha \circ T(x)) \leq 1,$$

or equivalently, $v(x)\, W_{\pi, T} f_\alpha(x) \in G$.

Thus $W_{\pi, T}$ is continuous at 0 and hence, by its inearity, on $FV_b(X, E)$.

Similarly, we can prove the following :

Proposition 3.3 — *Let E be a Hausdorff topological vector space andlet T be a selfmap on X. Then C_T is a composition operator on $FV_b(X, E)$ if and only if $V \leq V \circ T$, that is, for every $v \in V$ there exists $u \in V$ such that $v(x) \leq u \circ T(x)$ for all $x \in X$.*

Remark 3.4 : The condition (*b*) of Theorem 3.2 is not sufficient for the pair (π, T) to induce a weighted composition operator on $FV_0(X, E)$ as the following example shows :

Example 1 — Let $X = \mathbb{N}$ with the discrete topology, $E = \mathbb{R}$, and let $V = K^+(X)$. Then $FV_0(X, E) = c_0$, the space of all null sequences in R. Define $\pi(n) = 1$ and $T(n) = n_0$ for all $n \in \mathbb{N}$ and some $n_0 \in \mathbb{N}$. Then the pair (π, T) satisfies condition (*b*) of Theorem 3.2, but $W_{\pi, T}$ is not an operator on c_0. It is not even an into map, for if $f(n) = \frac{1}{n}$ then $f \in c_0$ but $W_{\pi, T} f \notin c_0$.

Therefore, we require some more conditions on the pair (π, T) so that $W_{\pi, T}$ is a weighted composition operator on $FV_0(X, E)$. Let $spt(v)$ denote the support of $v \in V$. In the following theorem, we shall present a necessary and sufficient condition for $W_{\pi, T}$ to be a weighted composition operator on $FV_0(X, E)$:

Theorem 3.5 — *Suppose E is a Hausdorff locally idempotent algebra and $\mathcal{N}$ is a base of neighbourhoods of 0 in E. Then the following conditions are sufficient for mappings $\pi : X \to E$ and $T : X \to X$ to induce a weighted composition operator on $FV_0(X, E)$: (a) for every $v \in V$ and $G \in \mathcal{N}$, there exist $u \in V$ and $H \in \mathcal{N}$ such that $v(x)\, p_G(\pi(x)\, t) < u \circ T(x)\, p_H(t)$ for all $x \in X$ and $t \in E$; and (b) for every $v \in V, G \in \mathcal{N}$ and compact subset K of X, the set $T^{-1}(K) \cap spt(v \cdot p_G \circ \pi)$ is a compact subset of X.*

Further, the above conditions are also necessary if E has unit element e, π is a semi-continuous and T is continuous.

PROOF : We may assume that $\mathcal{N}$ consists of closed, balanced, shrinkable, and idempotent sets.

Sufficient Part : Assume that conditions (a) and (b) are true. Then, by Theorem 3.2, $W_{\pi, T}$ is a weighted composition operator on $FV_b(X, E)$. We show that $FV_0(X, E)$ is invariant under $W_{\pi, T}$. For this let $f \in FV_0(X, E)$, $v \in V$ and $G \in \mathcal{N}$. By (*a*), we choose $u \in V$ and $H \in \mathcal{N}$ such that

$$v(x)\, p_G(\pi(x)t) < u \circ T(x)\, p_H(t) \qquad \dots (*)$$

for all $x \in X$ and $t \in E$. Now, given any $\varepsilon > 0$, there exists a compact subset K of X such that

$$u(x)\, p_H(f(x)) < \varepsilon \text{ for all } x \notin K.$$

Let $K_1 = T^{-1}(K) \cap spt(v \cdot p_G \circ \pi)$. Then K_1 is compact, by (*b*). If $x \notin K_1$, then either $x \notin T^{-1}(K)$ or $x \notin spt(u \cdot p_G \circ \pi)$, suppose first that $x \notin T^{-1}(K)$, then $T(x) \notin K$, and so $u \circ T(x)\, p_H(f \circ T(x)) < \varepsilon)$. Suppose first that $x \notin T^{-1}(K)$. Then $T(x) \notin K$, and so $u \circ T(x)$ $p_H(f \circ T(x)) < \varepsilon$. By (*), this implies that $v(x)\, p_G(\pi(x) f \circ T(x)) < \varepsilon$. If $x \notin spt(v \cdot p_G \circ \pi)$, then $v(x)\, p_G(\pi(x)) = 0$. Since p_G is submultiplicative, this implies

$$v(X)\, p_G(\pi(x) f \circ T(x)) \leq v(x)\, p_G(\pi(x))\, p_G(f \circ T(x)) = 0 < \varepsilon.$$

Thus $v \cdot W_{\pi, T} \in F_0(X, E)$. Since $v \in V$ was arbitrary, we conclude that $W_{\pi, T} f \in FV_0(X, E)$.

Necessary Part — Assume that E has a unit element e, π is upper semi-continuous, and T is continuous. Let $W_{\pi, T}$ be an operator on $FV_0(X, E)$. Then condition (*a*) follows in view of Theorem 3.2. To see that condition (*b*) holds, let $v \in V, G \in \mathcal{N}$ and K a compact subset of X be given. Define $f : X \to E$ as $f(x) = e$, if $x \in K$, and 0 if $x \notin K$. Then $f \in FV_0(X, E)$, and so the set

$$K_2 = \{x \in X : v(x)\, p_G(\pi(x) f \circ T(x)) \geq \varepsilon\}$$

is compact for every $\varepsilon > 0$. Since T is continuous and π is upper semi-continuous, we note that the set $K_1 = T^{-1}(K) \cap spt(v \cdot P_G \circ \pi)$ is cosed and is also contained in K_2. Hence, K_1 is compact and this completes the proof.

In a similar way, we can prove the following result :

Proposition 3.6 — *Let E be a Hausdorff topological vector space. Then the following conditions are sufficient for selfmap T on X to induce a composition operator on $FV_0(X, E)$:*

(a) $V \leq V \circ T$ and

(b) for every $v \in V$ and compact subset K of X, the set $T^{-1}(K) \cap spt(v)$ is compact. Further these conditions (a) and (b) are also necessary if T is continuous.

Remark 3.7 : Proposition 3.6 generalizes Proposition 1 of [11], Propositions 3.3 and 3.6 respectively generalizes Theorems 2.2 and 2.3 of [17], and Theorems 3.2 and 3.5 respectively generalizes Theorems 3.6 and 3.2 of [14], and Theorems 3.8 and 3.9 of [15] to a non-locally convex setting of weighted spaces of vector-valued functions. Also, Theorems 3.2 and 3.5 generalizes Theorem 3.2 of [7] to weighted spaces of vector-valued functions.

In the following results, we shall record characterizations of weighted composition operators $W_{\theta, T}$ on $FV_b(X, E)$ and $FV_0(X, E)$.

Theorem 3.8 — *Let E be a Hausdorff topological vector space, and let θ $\theta : X \to \mathbb{C}$ and $T : X \to X$ are two mappings. Then $W_{\theta, T}$ is a weighted composition operator on $FV_b(X, E)$ if and only if $V|\theta| \leq V \circ T$.*

PROOF : It follows from Theorem 3.2 with a slight modification in its proof.

Theorem 3.9 — *Let E be a Hausdorff topological vector space. Then the following conditions are sufficient for mappings : $\theta : X \to \mathbb{C}$ and $T : X \to X$ to induce a weighted composition operator on $FV_0(X, E)$*

(a) $V \cdot |\theta| \leq V \circ T$; and

(b) for every $v \in V$ and compact subset K of X, the set $T^{-1}(K) \cap spt(v \cdot |\theta|)$ is a compact subset of X. Further, the above conditions are also necessary if θ is upper semi-continuous and T is continuous.

PROOF : It follows from Theorem 3.5 with a slight modification in its proof.

Remark 3.10 : Theorems 3.8 and 3.9 generalizes respectively Theorems 3.5 and 3.6 of [15] to a non-locally convex setting of weighted spaces of vector valued functions, and also generalizes Theorem 3.1 of [7] to a general setting of weighted spaces of vector-valued functions.

Now we apply the above results to the cases when $V = S_c^+(X)$, $V = S_0^+(X)$ and $V = K^+(X)$.

Proposition 3.11 — *Let E be a Hausdorff locally idempotent algebra. Then every $\pi : X \to E$ induces a multiplication operator on $(F(X, E), k)$.*

PROOF : Let $\mathcal{N}$ be a base of neighbourhoods of 0 in E consisting of closed, balanced, idempotent, and shrinkable sets. We need to verify that condition (b) of Theorem 3.2 holds for $V = S_c^+(X)$. Let $v \in V$ and $G \in \mathcal{N}$. Choose a compact set $K \subset X$ with $v(x) = 0$ for all $x \in X \setminus K$. Let $\alpha = \sup\{p_G(\pi(x)) : x \in K\}$ and $\beta = \sup\{v(x) : x \in K\}$. Put $u = \beta\alpha\chi_k + 1$ and $H = G$. Then $u \in V$. Now for $x \in K$ and $t \in E$, we have

$$v(x)\, p_G(\pi(x)\, t) \leq \beta\, p_G(\pi(x))\, p_G(t) \leq \beta\alpha\, \chi_k(x)\, p_G(t) < u(x)\, p_H(t).$$

If $x \in X \setminus K$, the above inequality holds trivially (since $v(x) = 0$). Thus M_π is a multiplication operator on $FV_b(X, E) = (F(X, E), k)$.

Similarly, we can prove the following proposition :

Proposition 3.12 — Let E be a Hausdorff topological vector space. Then every $\theta : X \to \mathbb{C}$ induces a multiplication operator on $(F(X, E), k)$.

Proposition 3.13 : Let E be a Hausdorff topological space and let T be a continuous selfmap on X. Then C_T is a composition operator on $(F(X, E), k)$.

PROOF : In view of Proposition 3.3, we only need to verify that $V \le V \circ T$, where $V = \{\alpha \chi_k : \alpha \ge 0$ and $K \subset X, K$ compact$\}$. Let $v = \alpha \chi_K$, where K is a compact subset of X. Then there exists a compact subset F of X such that $K \subset T^{-1}(F)$. Let $u = \alpha \chi_F$. Then $u \in V$ and

$$v = \alpha \chi_K \le \alpha \chi_{T^{-1} F} = \alpha \chi_F \circ T = u \circ T.$$

This completes the proof of the Proposition.

Corollary 3.14 — Suppose E be a Hausdorff locally idempotent algebra (topological vector space). Let $\pi : X \to E$ $(\theta : X \to \mathbb{C})$ and let T be a continuous selfmap on X. Then $W_{\pi, T}$ $(W_{\theta, T})$ is a weighted composition operator on $(F(X, E), k)$.

Remark 3.15 : Corollary 3.14 generalizes Proposition 4.2 of [15] and Theorem 3.3 of [7] to a general setting of vector-valued functions.

Next, we remark that the Propositions 3.11, 3.12 and 3.13 need not hold for the space $FV_0(X, E)$ when $V = S_0^+(X)$ or $V = K^+(X)$. To see this, let us consider the following examples :

Example 2 — Let $X = \mathbb{R}^+ \setminus \{0\}$, $E = \mathbb{C}$ and $V = \{\alpha v : \alpha > 0\}$, where $v(x) = \frac{1}{x}$ for all $x \in X$. Let $\theta = \pi : X \to E$ be given by $\theta(x) = x^2$ for all $x \in X$. Then, for any $\alpha > 0$, we have $\alpha v(x) | \theta(x) | = \alpha x$ for all $x \in X$. Since each $u \in V$ is a bounded function, $v | \theta | \not\le u$ for every $u \in V$. Hence by Theorem 3.8, M_θ is not a multiplication operator on $(F_b(X, E), \beta_0)$. Further, if $T : X \to X$ is given by $T(x) = \frac{1}{x}$ for all $x \in X$, then it can be easily seen that $V \not\le V \circ T$. Therefore, according to Proposition 3.3, C_T is not a composition operator on $(F_b(X, E), \beta_0)$.

Example 3 — Let X, V and E be the same as in Example 1 so that $FV_0(X, E) = c_0$. If we define $\theta(x) = x$ for all $x \in X$, then $V | \theta | \not\le V$. Therefore, by Theorem 3.8, M_θ is not a multiplication operator on c_0. In fact, M_θ is not an even into map, for if, $f(x) = \frac{1}{x}$ for all $x \in X$, then $f \in c_0$ but $M_\theta f \notin c_0$. Similarly, if T is the map as defined in Example 1, then C_T is not a composotion operator on c_0.

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