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GENERALIZED F-INVEXITY AND DUALITY FOR PROGRAMMING PROBLEMS WITH SQUARE ROOT TERMS IN OBJECTIVES AND CONSTRAINTS

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In this paper, we introduce the concept of (F, ρ) - invexity for non-smooth Lipschitz functions. An example is provided to show that there exist functions that are (F, ρ)-invex but not (F, ρ)-convex. Two duals are proposed for multiobjective programming problem whose objectives as well as the constraints involve sum of non-smooth functions and terms involving square root of certain positive semi-definite quadratic forms. Duality results are established under generalized (F, ρ) - invexity assumptions.

Key Words : Multiobjective Programming; Non-smooth (F, ρ)-invexity; Duality

1. INTRODUCTION AND PRELIMINARIES

Under the assumption that means, variances and covariances of the random variables are known, Sinha[13] formulated a stochastic linear programming problem in the form of the following convex programming problem

(P1) Maximize $D^T X - (X^T BX)^{1/2}$

subject to $A_i X + (X^T B^i X)^{1/2} \leq b_i \;\; i = 1, 2, ..., m+1$

$X \geq 0,$

where $A_i, i = 1, 2, ..., m+1$ are $l \times (n+1)$ matrices, D, X are $(n+1) \times l$ matrices and B^i, $i = 0, 1, ..., m$ are $(n+1) \times (n+1)$ symmetric positive semidefinite matrices.

The particular case of the above problem in which only the coefficients of the objective function were random variables was stated as

(P2) Maximize $D^T X - (X^T BX)^{1/2}$

subject to $AX \leq b$

$X \geq 0.$

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The problem (P1) thus reduced to a nonlinear program where nonlinearity occurred only in the objective function. He established the duality results for (P2) under the assumption of boundedness on the primal constraint set. This assumption was relaxed by Bhatia[1], Mond[9], Chandra *et al.*[5], Mond and Smart[10], Bhatia and Tiwari[2], Zhang and Mond[15] studied various programming problems with square root terms only in the objective function and established duality results under different conditions on the functions involved.

In this paper, we consider a general form of the problem where the objectives as well as the constraints involve square roots of the positive semidefinite quadratic forms.

We concentrate on the following multiobjective optimization problem.

(VMP) Minimize $(\theta_1(x), \ldots, \theta_p(x))$

$$\text{subject to } G_j(x) \le 0,\ j = 1, 2, \ldots, m, \qquad \ldots (1.1)$$

where

$$\theta_i(x) = f_i(x) + (x^T B_i x)^{1/2},\ j = 1, 2, \ldots, p;$$

$$G_j(x) = g_j(x) + (x^T C_j x)^{1/2},\ j = 1, 2, \ldots, m;$$

and functions $f_i : R^n \to R$, $i = 1, 2, \ldots, p$; $g_j : R^n \to R, j = 1, 2, \ldots, m$; B_i, $i = 1, 2, \ldots, p$ and $C_j, j = 1, 2, \ldots, m$ are $n \times n$ symmetric positive semi-definite matrices. Duality results for (VMP) and its two duals are derived in the broad setting of (F, ρ)-invexity. The two duals of (VMP) studied in this paper are generalizations of the Mond-Weir dual problem[11].

Definition 1.1 — A feasible point $x^0 \in R^n$ is said to be efficient solution for (VMP) if there is no other feasible x such that

$$\theta_i(x) \le \theta_i(x^0) \text{ for all } i = 1, 2, \ldots, p,\ i \ne r$$

and

$$\theta_r(x) < \theta_r(x^0) \text{ for some } r \in \{1, 2, \ldots, p\}.$$

Definition 1.2 — A function $F : X \times X \times R^n \to R$ (where $X \subseteq R^n$) is sublinear if for any $x, x^0 \in X$,

$$F(x, x^0; a_1 + a_2) \le F(x, x^0; a_1) + F(x, x^0; a_2) \text{ for any } a_1, a_2 \in R^n \qquad \ldots (A)$$

$$F(x, x^0; \alpha a) \le \alpha F(x, x^0; a) \text{ for any } \alpha \in R, \alpha \ge 0 \text{ and } a \in R^n. \qquad \ldots (B)$$

From (A) and (B) it follows $F(x, x^0; 0) = 0$.

Let us consider a sublinear function F and the function $\phi : X \to R$ (where X is an open subset of R^n). We suppose ϕ is Lipschitz on X and regular (Clarke[7]) at x^0, an interior point of X.

Definition 1.3 — The function $\phi : X \to R$ is said to be (F, ρ)-invex with respect to η at $x^0 \in X$, if there exist functions $\eta, \psi : X \times X \to R^n$ and a real number ρ such that for all $x \in X$, we have

$$\phi(x) - \phi(x^0) \geq F(x, x^0; \xi^t \eta(x, x^0)) + \rho \| \psi(x, x^0) \|^2 \text{ for every } \xi \in \partial_\eta \phi(x^0)$$

where $\partial_\eta \phi(x^0)$ denotes the generalized η-gradient defined by

$$\partial_\eta \phi(x^0) = \{\xi \in R^n : \phi^0(x^0; \eta(x, x^0) \geq \xi^t \eta(x, x^0), \text{ for all } x \in X\}$$

and $\phi^0(x^0; \eta(x, x^0))$ is the generalized directional derivative of Clarke[7] defined by

$$\phi^0(x^0; \eta(x, x^0)) = \lim_{\lambda \downarrow 0} cdot; \sup_{y \to x^0} \frac{\phi(y + \lambda\eta(x, x^0)) - \phi(y)}{\lambda}.$$

The function ϕ is said to be strongly F-invex, F-invex or weakly F-invex with respect to η at x^0, according as $\rho > 0, \rho = 0, \rho < 0$.

Definition 1.4 — The function ϕ is said to be (F, ρ)-quasi invex with respect to η at $x^0 \in X$, if there exist functions $\eta, \psi : X \times X \to R^n$ and a real number ρ such that for all $x \in X$.

$$\phi(x) \leq \phi(x^0) \text{ implies } F(x, x^0; \xi^T \eta(x, x^0)) \leq -\rho \| \psi(x, x^0) \|^2 \text{ for every } \xi \in \hat{\partial}_\eta \phi(x^0)$$

Definition 1.5 — The function ϕ is said to be (F, ρ)-pseudo invex with respect to η at $x^0 \in X$, if there exist functions $\eta, \psi : X \times X \to R^n$ and a real number ρ such that for all $x \in X$ and for every $\xi \in \partial_\eta \phi(x^0)$,

$$F(x, x^0; \xi^T \eta(x, x^0)) \geq -\rho \| \psi(x, x^0) \|^2 \text{ implies } \phi(x) \geq \phi(x^0).$$

Definition 1.6 — The function ϕ is strictly (F, ρ)-quasi invex with respect to η at $x^0 \in X$, if there exists functions $\eta, \psi : X \times X \to R^n$ and a real number ρ such that for all $x \in X, x \neq x^0$,

$$\phi(x) \leq \phi(x^0) \text{ implies } F(x, x^0; \xi^T \eta(x, x^0)) < -\rho \| \psi(x, x^0) \|^2 \text{ for every } \xi \in \partial_\eta \phi(x^0).$$

Definition 1.7 — The function ϕ is strictly (F, ρ)-pseudo invex with respect to η at $x^0 \in X$, if there exists functions $\eta, \psi : X \times X \to R^n$ and a real number ρ such that for all $x \in X, x \neq x^0$,

$$F(x, x^0; \xi^T \eta(x, x^0)) \geq -\rho \| \psi(x, x^0) \|^2 \text{ implies } \phi(x) > \phi(x^0).$$

Every (F, ρ)-invex function with respect to η is both (F, ρ)-quasi invex and (F, ρ)-pseudo invex. Also there exist non-smooth functions which are (F, ρ)-invex with respect to some η but not (F, ρ)-convex (e.g. Bhatia and Jain[3]) as can be seen from the following example.

Example 1.1 — Define the function $f : [-1, 1] \to R$ by

$$f(x) = \begin{cases} x^3 + x : -1 \leq x \leq 0 \\ 2x \quad : \ 0 \leq x \leq 1. \end{cases}$$

Here Clarke's generalized gradient of f at $x^0 = 0$ is

$$\partial f(0) = \{\xi : 1 \leq \xi \leq 2\}.$$

For all $\xi \in \partial f(0)$ and $-1 \leq x \leq 0$, the function f is not (F, ρ)-convex at $x^0 = 0$ and $\rho = 1$ with respect to the sublinear function $F(x, 0; \xi)$ defined as

$$F(x, 0; \xi) = \xi x^2$$

and $d(x, x^0) = (x - x^0)^{1/2}$. But the function f is (F, ρ)-invex with respect to η at $x^0 = 0$ and $\rho = 1$ where the functions $\eta, \psi : X \times X \rightarrow R$ are defined as

$$\eta(x, x^0) = x + x^0 \text{ and } \psi(x, x^0) = (x + x^0 - 2)^{1/2}.$$

The generalized η-gradient of f at x^0 is

$$\partial_\eta f(0) = \begin{cases} \xi \geq 1, -1 \leq x \leq 0 \\ \xi \leq 1, 0 \leq x \leq 1. \end{cases}$$

Remark 1.1 : If the objective function of an optimization problem is —

Minimize

$$\theta(x) = f(x) + (x^T Bx)^{1/2}$$

where $f(x)$ is as defined in Example 1.1 above, then we have an optimization problem for which the new results established in this paper apply but not the earlier results existing in literature.

We need the following results in the sequel.

Lemma 1.1 (Chankong[6]) — x^0 is an efficient solution for (VMP) if and only if x^0 solves

$P_r(x^0)$ Minimize $\theta_r(x)$

subject to $\theta_k(x) \leq \theta_k(x^0)$, $k \neq r, k = 1, 2, ..., p$

$G_j(x) \leq 0$, $j = 1, 2, ..., m$

for each $r = 1, 2, ..., p$.

Consider the optimization problem

(P3) Minimize $\phi(x)$

subject to $G_j(x) \leq 0$, $j = 1, 2, ..., m$,

where $\phi : R^n \rightarrow R$, $G_j : R^n \rightarrow R$, $j = 1, 2, ..., m$ are locally Lipschitz and regular.

The problem (P3) is related to the following problem for $s \in R^m$.

(P4) Minimize $\phi(x)$

subject to $G_j(x) \leq s_j$, $j = 1, 2, ..., m$.

Definition 1.12 (Tanaka *et al.*[14]) — Problem (P3) is said to be calm at $x^0 \in R^n$ if for all sequences $x^k \rightarrow x^0$ with $s^k \rightarrow 0$ such that x^k is feasible for (P4) with $s = s^k$, we have

$\frac{\phi(x) - \phi(x^k)}{\| s^k \|} \leq M$ for some constant M.

Lemma 1.2 (Craven and Mond[8]) — Let $\phi(x) = (x^T Bx)^{1/2}$. Then $\phi(x)$ is convex and $w \in \partial_\eta \phi(x)$ if and only if $w = Bz, z^T Bz \leq 1, x^T Bz = (x^T Bx)^{1/2}$.

Lemma 1.3 (Bhatia and Jain[4]) — Let $\phi(x) = (x^T Bx)^{1/2}$. Then $\phi(x)$ is locally Lipschitz.

Remark : When $x = 0$, $\phi'(0; d) = (d^T Bd)^{1/2}$ for any direction d, and when $x \neq 0$, the function $\phi(x)$ is convex differentiable. Hence $\phi(x)$ is quasi-differentiable. Moreover, the function $\phi(x)$ is regular.

Theorem 1.1 *(Schechter[12]) (Fritz-John type necessary condition) — Let x^* is an efficient solution for (VMP). Then there exist $\alpha_i^* \in R, i = 1, 2, ..., p; \lambda_j^* \in R, j = 1, 2, ..., m$ such that*

$$\lambda_j^* G_j(x^*) = 0, \; j = 1, 2, ..., m \qquad ... (1.2)$$

$$0 \in \sum_{i=1}^{p} \alpha_i^* \partial_\eta \theta_i(x^*) + \sum_{j=1}^{m} \lambda_j^* \partial_n G_j(x^*), \qquad ... (1.3)$$

$$(\alpha_1^*, ..., \alpha_p^*, \lambda_1^*, ..., \lambda_m^*) \geq 0 \qquad ... (1.4)$$

$$(\alpha_1^*, ..., \alpha_p^*, \lambda_1^*, ..., \lambda_m^*) \neq 0. \qquad ... (1.5)$$

2. Duality I

In this section, we obtain weak and strong duality relations between (VMP) and the following nondifferentiable multiobjective programming problem :

(VMD1) Maximize $(\phi_1(\alpha, u, \lambda, z, v), ..., \phi_p(\alpha, u, \lambda, z, v))$

subject to

$$-\left[\sum_{i=1}^{p} \alpha_i z^T B_i + \sum_{j=1}^{m} \lambda_j v^T C_j \right] \in \sum_{i=1}^{p} \alpha_i \partial_\eta f_i(u) + \sum_{j=1}^{m} \lambda_j \partial_\eta g_j(u) \qquad ... (2.1)$$

$$\lambda_j [g_j(u) + v^T C_j u] \geq 0, \; j = 1, 2, ..., m \qquad ... (2.2)$$

$$z^T B_i z \leq 1, \; i = 1, 2, ..., p, \qquad ... (2.3)$$

$$v^T C_j v \leq 1, \; j = 1, 2, ..., m, \qquad ... (2.4)$$

$$(\alpha_1, ..., \alpha_p, \lambda_1, ..., \lambda_m) \geq 0 \qquad ... (2.5)$$

and $$(\alpha_1, ..., \alpha_p, \lambda_1, ..., \lambda_m) \neq 0, \qquad ... (2.6)$$

where $$\phi_i(\alpha, u, \lambda, z, v) = f_i(u) + z^T B_i u.$$

Theoem 2.1 (*Weak Duality*) — *Assume that for all feasible x for (VMP) and all feasible $(\alpha, u, \lambda, z, v)$ for (VMD1),*

(a1) f_i is (F, ρ_{1i})-invex at u with respect to same η, $i = 1, 2, \ldots, p$;

(a2) g_j is (F, ρ_{2j})-invex at u with respect to same η, $j = 1, 2, \ldots, m$;

(a3) $z^T B_i(\cdot)$ is (F, ρ_{3i})-invex at u with respect to same η, $i = 1, 2, \ldots, p$;

(a4) $v^T C_j(\cdot)$ is (F, ρ_{4j})-invex at u with respect to same η, $j = 1, 2, \ldots, m$;

(a5) $\left(\sum_{i=1}^{p} \alpha_i \rho_{1i} + \sum_{j=1}^{m} \lambda_j \rho_{2j} + \sum_{i=1}^{p} \alpha_i \rho_{3i} + \sum_{j=1}^{m} \lambda_j \rho_{4j}\right) \geq 0$; *and*

(a6) $\alpha_i > 0$ for all $i = 1, 2, \ldots, p$.

Then the following cannot hold

$$\begin{bmatrix} \theta_i(x) \leq \phi_i(\alpha, u, \lambda, z, v) & \text{for all } i = 1, 2, \ldots, p, i \neq r \\ \theta_r(x) < \phi_r(\alpha, u, \lambda, z, v) & \text{for some } r. \end{bmatrix} \qquad \ldots (2.7)$$

PROOF : Since $(\alpha, u, \lambda, z, v)$ is feasible for (VMD1), we have

$$-\left[\sum_{i=1}^{p} \alpha_i z^T B_i + \sum_{j=1}^{m} \lambda_j v^T C_j\right] = \sum_{i=1}^{p} \alpha_i \xi_i + \sum_{j=1}^{m} \lambda_j \beta_j$$

where $\xi_i \in \partial_\eta f_i(u)$, $i = 1, 2, \ldots, p$; $\beta_j \in \partial_\eta g_j(u), j = 1, 2, \ldots, m$.

That is,

$$0 = \sum_{i=1}^{p} \alpha_i z^T B_i \eta(x, u) + \sum_{j=1}^{m} \lambda_j v^T C_j \eta(x, u) + \sum_{i=1}^{p} \alpha_i \xi_i^T \eta(x, u) + \sum_{j=1}^{m} \lambda_j \beta_j^T \eta(x, u).$$

This implies that

$$0 = F\left(x, u; \sum_{i=1}^{p} \alpha_i z^T B_i \eta(x, u) + \sum_{j=1}^{m} \lambda_j v^T C_j \eta(x, u) + \sum_{i=1}^{p} \alpha_i \xi_i^T \eta(x, u) + \sum_{j=1}^{m} \lambda_j \beta_j^T \eta(x, u)\right).$$

Then subadditivity of F implies

$$0 \leq \sum_{i=1}^{p} \alpha_i F(x, u; z^T B_i \eta(x, u)) + \sum_{j=1}^{m} \lambda_j F(x, u; v^T C_j \eta(x, u)) + \sum_{i=1}^{p} \alpha_i F(x, u; \xi_i^T \eta(x, u))$$

$$+ \sum_{j=1}^{m} \lambda_j F(x, u; \beta_j^T \eta(x, u)). \qquad \ldots (2.8)$$

From assumptions (a1)-(a4), we have

$$f_i(x)-f_i(u) \geq F(x, u; \xi_i^T \eta(x, u)) + \rho_{1i} \| \psi(x, u)(x, u) \|^2, \; i = 1, 2, \ldots, p,$$

$$g_j(x)-g_j(u) \geq F(x, u; \beta_j^T \eta(x, u)) + \rho_{2j} \| \psi(x, u) \|^2, \; j = 1, 2, \ldots, m,$$

$$z^T B_i(x-u) \geq F(x, u; z^T B_i \eta(x, u)) + \rho_{3i} \| \psi(x, u) \|^2, \; i = 1, 2, \ldots, p$$

and

$$v^T C_j (x-u) \geq F(x, u; v^T C_j \eta(x, u)) + \rho_{4j} \| \psi(x, u) \|^2, \; j = 1, 2, \ldots, m.$$

Because of (2.5) and (2.6), the above inequalities imply that

$$\sum_{i=1}^{p} \alpha_i f_i(x) - \sum_{i=1}^{p} \alpha_i f_i(u) \geq \sum_{i=1}^{p} \alpha_i F(x, u; \xi_i^T \eta(x, u)) + \sum_{i=1}^{p} \alpha_i \rho_{1i} \| \psi(x, u) \|^2 \qquad \ldots (2.9)$$

$$\sum_{j=1}^{m} \lambda_j g_j(x) - \sum_{j=1}^{m} \lambda_j g_j(u) \geq \sum_{j=1}^{m} \lambda_j F(x, u; \beta_j^T \eta(x, u)) + \sum_{j=1}^{m} \lambda_j \rho_{2j} \| \psi(x, u) \|^2 \qquad \ldots (2.10)$$

$$\sum_{i=1}^{p} \alpha_i z^T B_i(x-u) \geq \sum_{i=1}^{p} \alpha_i F(x, u; z^T B_i \eta(x, u)) + \sum_{i=1}^{p} \alpha_i \rho_{3i} \| \psi(x, u) \|^2 \qquad \ldots (2.11)$$

$$\sum_{j=1}^{m} \lambda_j v^T C_j (x-u) \geq \sum_{j=1}^{m} \lambda_j F(x, u; v^T C_j \eta(x, u)) + \sum_{j=1}^{m} \lambda_j \rho_{4j} \| \psi(x, u) \|^2. \qquad \ldots (2.12)$$

Substituting (2.9)-(2.12) into (2.8), we obtain

$$0 \leq \sum_{i=1}^{p} \alpha_i f_i(x) - \sum_{i=1}^{p} \alpha_i f_i(u) + \sum_{j=1}^{m} \lambda_j g_j(x) - \sum_{j=1}^{m} \lambda_j g_j(u)$$

$$+ \sum_{i=1}^{p} \alpha_i z^T B_i(x-u) + \sum_{j=1}^{m} \lambda_j v^T C_j(x-u)$$

$$- \left(\sum_{i=1}^{p} \alpha_i \rho_{1i} + \sum_{j=1}^{m} \lambda_j \rho_{2j} + \sum_{i=1}^{p} \alpha_i \rho_{3i} + \sum_{j=1}^{m} \lambda_j \rho_{4j} \right) \| \psi(x, u) \|^2.$$

Using assumption (a5), Schwarz's inequality, (2.3) and (2.4), the above inequality implies

$$0 \leq \sum_{i=1}^{p} \alpha_i f_i(x) - \sum_{i=1}^{p} \alpha_i f_i(u) + \sum_{j=1}^{m} \lambda_j g_j(x) - \sum_{j=1}^{m} \lambda_j g_j(u)$$

$$- \sum_{i=1}^{p} \alpha_i z^T B_i u + \sum_{i=1}^{p} \alpha_i (x^T B_i x)^{1/2} - \sum_{j=1}^{m} \lambda_j v^T C_j u + \sum_{j=1}^{m} \lambda_j (x^T C_j x)^{1/2}.$$

Using (1.1) and (2.2) in above inequality, we obtain

$$0 \leq \sum_{i=1}^{p} \alpha_i (f_i(x) + (x^T B_i x)^{1/2}) - \sum_{i=1}^{p} \alpha_i (f_i(u) + z^T B_i u)$$

$$\sum_{i=1}^{p} \alpha_i (f_i(x) + (x^T B_i x)^{1/2}) \geq \sum_{i=1}^{p} \alpha_i (f_i(u) + z^T B_i u).$$

Since $\alpha_i > 0$ for all $i = 1, 2, ..., p$, hence (2.7) cannot hold.

Theorem 2.2 (*Weak Duality*) — *Let x be feasible for (VMP) and $(\alpha, u, \lambda, z, v)$ be feasible for (VMD 1). Let the functions f_i, $i = 1, 2, ..., p$; $g_j, j = 1, 2, ..., m$; $z^T B_i(\cdot)$, $i = 1, 2, ..., p$; $v^T C_j(\cdot)$, $j = 1, 2, ..., m$ be F-convex and suppose that $\alpha_i > 0$ for all $i = 1, 2, ..., p$. Then (2.7) cannot hold.*

PROOF : Proof follows by proceeding in the same manner as in Theorem 2.1 with appropriate modifications and is hence omitted.

Theorem 2.3 (*Weak Duality*) — *Let x be feasible for (VMP) and $(\alpha, u, \lambda, z, v)$ be feasible for (VMD1) and $\lambda^T g(\cdot) + \lambda^T v^T C_j(\cdot)$ is (F, ρ)-quasi invex with respect to η at u and if any one of the following two holds*

(b1) $\alpha_i > 0$ for all $i = 1, 2, ..., p$ and $\alpha^T f(\cdot) + \alpha^T z^T B_i(\cdot)$ is (F, ρ^*)-pseudo invex with respect to η at u with $\rho + \rho^* \geq 0$ and

(b2) $\alpha^T f(\cdot) + \alpha^T z^T B_i(\cdot)$ is strictly (F, ρ^*)-pseudo invex with respect to η at u with $\rho + \rho^* > 0$.

Then (2.7) cannot hold.

PROOF : Since x is feasible for (VMP) and $(\alpha, u, \lambda, z, v)$ is feasible for (VMD1), we have from (1.1) and (2.2),

$$\lambda_j [g_j(x) + (x^T C_j x)^{1/2}] \leq \lambda_j [g_j(u) + v^T C_j u].$$

Using (2.5) and (2.6), we have

$$\lambda_j [g_j(x) + x^T C_j v] \leq \lambda_j [g_j(u) + v^T C_j u]. \qquad ... (2.13)$$

In view of the hypothesis that $\lambda^T g(\cdot) + \lambda^T v^T C_j(\cdot)$ is (F, ρ)-quasi invex with respect to η, therefore from (2.13), it follows that

$$F(x; u; (\lambda^T \beta + \lambda^T v^T C_j)\, \eta(x, u)) \leq -\rho \,\| \psi(x, u) \|^2 \text{ where } \beta \in \partial_\eta g(u). \qquad \text{... (2.14)}$$

From feasibility of $(\alpha, u, \lambda, z, v)$ for (VMD1), we have

$$F(x, u; (\alpha^T \xi + \lambda^T \beta + \lambda^T v^T C_j + \alpha^T z^T B_i)\, \eta(x, u)) = 0 \text{ where } \xi \in \partial_\eta f(u).$$

Using the sublinearity of F, we get

$$F(x, u; (\alpha^T \xi + \alpha^T z^T B_i)\eta (x, u)) + F(x, u; (\lambda^T \beta + \lambda^T v^T C_j)\eta (x, u)) \geq 0. \qquad \text{... (2.15)}$$

Using (2.14), we have

$$F(x, u; (\alpha^T \xi + \alpha^T z^T B_i)\eta (x, u)) \geq \rho \,\| \psi (x, u) \|^2$$

In case $(b1)$ holds, then using $\rho + \rho^* \geq 0$ in the above inequality, we have

$$F(x, u; (\alpha^T \xi + \alpha^T z^T B_i)\eta (x, u)) \geq -\rho^* \,\| \psi(x, u) \|^2 \qquad \text{... (2.16)}$$

(F, ρ^*)-pseudo invexity of $\alpha^T f(\cdot) + \alpha^T z^T B_i(\cdot)$ with respect to η at u implies

$$\alpha^T (f(x) + \alpha^T z^T B_i\, x \geq \alpha^T f(u) + \alpha^T z^T B_i\, u.$$

Since $\alpha_i > 0$ for all $i = 1, 2, ..., p$, therefore (2.7) cannot hold.

In case $(b2)$ holds, from (2.16) and using strict (F, ρ^*)-pseudo invexity of $\alpha^T f(\cdot) + \alpha^T z^T B_i(\cdot)$ with respect to η at u, we have

$$\alpha^T f(x) + \alpha^T z^T B_i\, x > \alpha^T f(u) + \alpha^T z^T B_i u$$

Hence, (2.7) again cannot hold.

Corollary 2.1 — Let x^o be (VMP) feasible and $(\alpha^o, u^o, \lambda^o, z^o, v^o)$ be (VMDI) feasible such that $\theta(x^o) = \phi(\alpha^o, u^o, \lambda^o, z^o, v^o)$. If Theorem 2.1, 2.2 or 2.3 holds between (VMP) and (VMDI), then x^o is efficient for (VMP) and $(\alpha^o, u^o, \lambda^o, z^o, v^o)$ is efficient for (VMDI).

Theorem 2.4 *(Strong Duality) — Let x^o be an efficient solution for (VMP). Then there exists $\alpha^o \in R^p, \lambda^o \in R^m, z^o \in R^n, v^o \in R^n$ such that $(\alpha^o, u^o, \lambda^o, z^o, v^o)$ is feasible for (VMDI) and the respective values of the objective functions of (VMP) and (VMDI) are equal. Furthermore, if the assumptions of Theorem 2.1, 2.2 or 2.3 are satisfied, then $(\alpha^o, x^o, \lambda^o, z^o, v^o)$ is efficient for (VMD1).*

PROOF : Since x^0 is efficient for (VMP), from Lemma 1.1, x^0 solves $P_r(x^0)$ for all $r = 1, 2, ..., p$.

By Theorem 1.1, there exist $\alpha^0 \in R^p, \lambda^0 \in R^m$ such that

$$\lambda_j^0\, [g_j(x^0) + (x^{0T} C_j x^0)^{1/2}] = 0,\; j = 1, 2, ..., m, \qquad \text{... (2.17)}$$

$$0 \in \sum_{i=1}^{p} \alpha_i^0 \partial_\eta \theta_i(x^0) + \sum_{j=1}^{m} \lambda_j^0 \partial_\eta G_j(x^0) \quad \ldots (2.18)$$

$$(\alpha_1^0, \ldots, \alpha_p^0, \lambda_1^0, \ldots, \lambda_m^0) \geq 0 \quad \ldots (2.19)$$

$$(\alpha_1^0, \ldots, \alpha_p^0, \lambda_1^0, \ldots, \lambda_m^0) \neq 0. \quad \ldots (2.20)$$

Condition (2.18) implies that

$$0 = \sum_{i=1}^{p} \alpha_i^0 \xi_i^0 + \sum_{i=1}^{p} \alpha_i^0 w_i^0 + \sum_{j=1}^{m} \lambda_j^0 \rho_j^0 + \sum_{j=1}^{m} \lambda_j^0 \eta_j^0$$

where

$$\xi_i^0 \in \partial_\eta f_i(x^0),\ i = 1, 2, \ldots, p;\ w_i^0 \in \partial_\eta ((x^{0T} B_i x^0)^{1/2}),\ i = 1, 2, \ldots, p;$$

$$\rho_j^0 \in \partial_\eta g_j(x^0),\ j = 1, 2, \ldots, m;\ \eta_j^0 \in \partial_\eta ((x^{0T} C_j x^0)^{1/2}),\ j = 1, 2, \ldots, m.$$

Therefore

$$-\sum_{i=1}^{p} \alpha_i^0 w_i^0 - \sum_{j=1}^{m} \lambda_j^0 \eta_j^0 = \sum_{i=1}^{p} \alpha_i^0 \xi_i^0 + \sum_{j=1}^{m} \lambda_j^0 \rho_j^0 . \quad \ldots (2.21)$$

As $w_i^0 \in \partial_\eta ((x^{0T} B_i^0 x^0)^{1/2})$ and $\eta_j^0 \in \partial_\eta ((x^{0T} C_j x^0)^{1/2})$, it follows from Lemma 1.2 that there exist $z^0 \in R^n$, $v^0 \in R^n$ respectively such that

$$w_i^0 = B_i z^0,\ z^{0T} B_i z^0 \leq 1,\ x^{0T} B_i z^0 = (x^{0T} B_i x^0)^{1/2} \quad \ldots (2.22)$$

$$\eta_j^0 = C_j v^0,\ v^{0T} C_j v^0 \leq 1,\ v^{0T} C_j v^0 = (x^{0T} C_j x^0)^{1/2}. \quad \ldots (2.23)$$

In view of (2.17) and (2.19)-(2.23), we have $(\alpha^0, x^0, \lambda^0, z^0, v^0)$ is (VMD 1) feasible. Also from (2.22), it follows that the values of the objective functions of (VMP) and (VMD 1) are equal.

If the assumptions of Theorem 2.1, 2.2 or 2.3 are satisfied, then using the equality of the two objectives and Corollary 2.1, we see that x^0 and $(\alpha^0, x^0, \lambda^0, z^0, v^0)$ are respectively efficient for (VMP) and (VMD 1).

3. Duality II

We now introduce dual problem (VMD2) for the problem (VMP) and establish duality results (VMD2) Maximize $(\phi_1(\alpha, u, \lambda, z, v), \ldots, \phi_p(\alpha, u, \lambda, z, v))$

subject to

$$-z^T B_k - \sum_{\substack{i=1\\ i\neq k}}^{p} \alpha_i z^T B_i - \sum_{j=1}^{m} \lambda_j v^T C_j \in \partial_\eta f_k(u) + \sum_{\substack{i=1\\ i\neq k}}^{p} \alpha_i \partial_\eta f_i(u) + \sum_{j=1}^{m} \lambda_j \partial_\eta g_j(u) \quad \ldots (3.1)$$

$$\lambda_j [g_j(u) + v^T C_j u] \geq 0,\ j = 1, 2, \ldots, m, \quad \ldots (3.2)$$

$$z^T B_i z \leq 1,\ i = 1, 2, \ldots, p, \quad \ldots (3.3)$$

$$v^T C_j v \leq 1,\ j = 1, 2, \ldots, m, \quad \ldots (3.4)$$

$$(\alpha_1, \ldots, \alpha_p, \lambda_1, \ldots, \lambda_m) \geq 0 \quad \ldots (3.5)$$

and

$$(\alpha_1, \ldots, \alpha_p, \lambda_1, \ldots, \lambda_m) \neq 0. \quad \ldots (3.6)$$

Theorem 3.1 (*Weak Duality*) — *Let x be feasible for (VMP) and $(\alpha, u, \lambda, z, v)$ be feasible for (VMD2). If*

(i) f_i is (F, ρ_{1i})-invex at u with respect to same η for $i = 1, 2, \ldots, p$;

(ii) g_j is (F, ρ_{2j})-invex at u with respect to same η for $j = 1, 2, \ldots, m$;

(iii) $z^T B_i(\cdot)$ is (F, ρ_{3i})-invex at u with respect to same η for $i = 1, 2, \ldots, p$;

(iv) $v^T C_j(\cdot)$ is (F, ρ_{4j})-invex at u with respect to same η for $j = 1, 2, \ldots, m$; and

(v) $$\left(\rho_{1k} + \sum_{\substack{i=1\\ i\neq k}}^{p} \alpha_i \rho_{1i} + \sum_{j=1}^{m} \lambda_j \rho_{2j} + \rho_{3i} + \sum_{\substack{i=1\\ i\neq k}}^{p} \alpha_i \rho_{3i} + \sum_{j=1}^{m} \rho_{4j} \right) \geq 0$$

then (2.7) cannot hold.

PROOF : Since $(\alpha, u, \lambda, z, v)$ is feasible for (VMD2), we have

$$-z^T B_k - \sum_{\substack{i=1\\ i\neq k}}^{p} \alpha_i z^T B_i - \sum_{j=1}^{m} \lambda_j v^T C_j = \xi_k + \sum_{\substack{i=1\\ i\neq k}}^{p} \alpha_i \xi_i + \sum_{j=1}^{m} \lambda_j \beta_j$$

where

$$\xi_i \in \partial_\eta f_i(u),\ i = 1, 2, \ldots, p;\ \beta_j \in \partial_\eta g_j(u),\ j = 1, 2, \ldots, m.$$

That is,

$$0 = \xi_k^T \eta(x,u) + \sum_{\substack{i=1 \\ i\neq k}}^{p} \alpha_i \xi_i^T \eta(x,u) + \sum_{\substack{i=1 \\ i\neq k}}^{p} \alpha_i z^T B_i \eta(x,u) + \sum_{j=1}^{m} \lambda_j \beta_j^T \eta(x,u)$$

$$+ \sum_{\substack{j=1 \\ i\neq k}}^{m} \lambda_j v^T C_j \eta(x,u) + z^T B_k.$$

Then subadditive property of F implies

$$0 \leq F(x,u;\xi_k{}^T\eta) + \sum_{\substack{i=1 \\ i\neq k}}^{p} \alpha_i F(x,u;\xi_i{}^T\eta) + F(x,u;z^T B_k \eta)$$

$$+ \sum_{\substack{i=1 \\ i\neq k}}^{p} \alpha_i F(x,u;z^T B_i \eta) + \sum_{j=1}^{m} \lambda_j F(x,u;\beta_j^T \eta) + \sum_{j=1}^{m} \lambda_j F(x,u;v^T C_j \eta). \quad \ldots (3.7)$$

In view of assumptions (i)-(iv) and (3.5) and (3.6), we have

$$f_k(x) - f_k(u) \geq F(x,u;\xi_k^T \eta) + \rho_{1k} \| \psi(x,u) \|^2$$

$$\sum_{\substack{i=1 \\ i\neq k}}^{p} \alpha_i f_i(x) - \sum_{\substack{i=1 \\ i\neq k}}^{p} \alpha_i f_i(u) \geq \sum_{\substack{i=1 \\ i\neq k}}^{p} \alpha_i F(x,u;\xi_i^T \eta) + \sum_{\substack{i=1 \\ i\neq k}}^{p} \alpha_i \rho_{1i} \| \psi(x,u) \|^2$$

$$\sum_{j=1}^{m} \lambda_j g_j(x) - \sum_{j=1}^{m} \lambda_j g_j(u) \geq \sum_{j=1}^{m} \lambda_j F(x,u;\beta_j^T \eta) + \sum_{j=1}^{m} \lambda_j \rho_{2j} \| \psi(x,u) \|^2$$

$$\sum_{\substack{i=1 \\ i\neq k}}^{p} \alpha_i z^T B_i (x-u) \geq \sum_{\substack{i=1 \\ i\neq k}}^{p} \alpha_i F(x,u;z^T B_i \eta) + \sum_{\substack{i=1 \\ i\neq k}}^{p} \alpha_i \rho_{3i} \| \psi(x,u) \|^2$$

$$z^T B_k (x-u) \geq F(x,u;z^T B_k \eta) + \rho_{3i} \| \psi(x,u) \|^2$$

$$\sum_{j=1}^{m} \lambda_j v^T C_j (x-u) \geq \sum_{j=1}^{m} \lambda_j F(x,u;v^T C_j \eta) + \sum_{j=1}^{m} \lambda_j \rho_{4j} \| \psi(x,u) \|^2.$$

Using these inequalities in (3.7), we obtain

$$0 \leq f_k(x) - f_k(u) + \sum_{i=1}^{p} \alpha_i f_i(x) - \sum_{i=1}^{p} \alpha_i f_i(u) + \sum_{j=1}^{m} \lambda_j g_j(x) -$$

$$\sum_{j=1}^{m} \lambda_j g_j(u) + \sum_{i=1}^{p} \alpha_i z^T B_i(x-u) + z^T B_k(x-u) + \sum_{j=1}^{m} \lambda_j v^T C_j(x-u) -$$

$$\left(\rho_{1k} + \sum_{i=1}^{p} \alpha_i \rho_{1i} + \sum_{j=1}^{m} \lambda_j \rho_{2i} + \rho_{3k} + \sum_{i=1}^{p} \alpha_i \rho_{3i} + \sum_{j=1}^{m} \lambda_j \rho_{4j} \right) \| \psi(x, u) \|^2$$

Using (1.1), (3.2), (3.5), (3.6), assumption (v) and Schwarz's inequality, the above inequality implies

$$0 \leq [f_k(x) + (x^T B_k x)^{1/2}] - [f_k(u) + z^T B_k u] + \sum_{\substack{i=1 \\ i \neq k}}^{p} \alpha_i [f_i(x) + (x^T B_i x)^{1/2}]$$

$$- \sum_{\substack{i=1 \\ i \neq k}}^{p} \alpha_i [f_i(u) + z^T B_i u].$$

This implies,

$$\sum_{\substack{i=1 \\ i \neq k}}^{p} \alpha_i [f_i(u) + z^T B_i u] + [f_k(u) + z^T B_k u] \leq [f_k(x) + (x^T B_k x)^{1/2}] + \sum_{\substack{i=1 \\ i \neq k}}^{p} \alpha_i [f_i(x) + (x^T B_i x)^{1/2}].$$

Hence (2.7) cannot hold.

Theorem* 3.2 (*Strong Duality*) — *Let x^0 be an efficient solution for (VMP) and let the problem (VMP) be calm at x^0 in the sense of Clarke. Then there exists $\alpha^0 \in R^p, \lambda^0 \in R^m, z^0 \in R^n, v^0 \in R^n$ such that $(\alpha^0, x^0, \lambda^0, z^0, v^0)$ is feasible for (VMD2). If the assumptions of Theorem 3.1 are satisfied, then $(\alpha^0, x^0, \lambda^0, z^0, v^0)$ is efficient for (VMD2).

PROOF : Since x^0 is efficient for (VMP), from Lemma 1.1, x^0 solves $P_r(x^0)$ for all r = 1, 2, ..., p and the problem $P_r(x^0)$ is calm at x^0 (Clarke[7]) there exist $\alpha^0 \in R^p, \lambda^0 \in R^m$ such that

$$0 \in \partial_\eta \theta_k(x^0) + \sum_{\substack{i=1 \\ i \neq k}}^{p} \alpha_i^0 \partial_\eta \theta_i(x^0) + \sum_{j=1}^{m} \lambda_j^0 \partial_\eta G_j(x^0) \quad \text{... (3.8)}$$

$$\alpha_i^0 \geq 0, \alpha_i^0 \neq 0, \ i = 1, 2, ..., p, \quad \text{... (3.9)}$$

$$\lambda_j^0 \geq 0, \lambda_j^0 \neq 0, \ j = 1, 2, ..., m, \quad \text{... (3.10)}$$

$$G_j(x^0) \leq 0, \ i = 1, 2, ..., m \quad \text{... (3.11)}$$

and

$$\lambda_j^0 G_j(x^0) = 0, \; j = 1, 2, \ldots, m. \qquad \ldots (3.12)$$

Condition (3.8) can be written as

$$0 \in \partial_\eta \theta_k(x^0) + \partial_\eta ((x^{0T} B_k x^0)^{1/2}) + \sum_{i=1}^{p} \alpha_i^0 \partial_\eta f_i(x^0)$$

$$+ \sum_{i=1}^{p} \alpha_i^0 \partial_\eta ((x^{0T} B_i x^0)^{1/2}) + \sum_{j=1}^{m} \lambda_i^0 \partial_\eta g_j(x^0)$$

$$+ \sum_{j=1}^{m} \lambda_j^0 \partial_\eta ((x^{0T} C_j x)^{1/2}). \qquad \ldots (3.13)$$

From (3.9) to (3.13) and Lemma 1.2, there exist $z^0 \in R^n, v^0 \in R^n$ such that $(\alpha^0, x^0, \lambda^0, z^0, v^0)$ is (VMD2) feasible. Furthermore, if the assumptions of Theorem 3.1 are true, then $(\alpha^0, x^0, \lambda^0, z^0, v^0)$ is efficient for (VMD2).

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