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A NOTE ON ARCWISE CONNECTED AND RELATED FUNCTIONS

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In this paper, arcwise connected sets are characterized in terms of arcwise combinations of more than two points. Functions related to arcwise connected functions namely strictly arcwise connected functions, semistrictly arcwise connected functions and arcwise affine functions are defined and certain relations and properties of these functions are studied. Necessary and sufficient conditions involving arcwise connected functions, strictly arcwise connected functions and semistrictly arcwise connected functions are established on the lines of the well-known Jensen's inequality for convex functions.

Key Words : Convexity; Arcwise Connectedness; Strict Arcwise Connectedness; Semistrict Arcwise Connectedness, Arcwise Affine Functions; Jensen's Inequality

1. INTRODUCTION

Avriel[1] generalized the concept of convex sets and functions to arcwise connectedness by replacing the line segment joining two points by a continuous arc on the lines suggested by Ortega and Rheinboldt[2]. Certain local-global minimum properties were discussed for generalized arcwise connected functions by Avriel and Zang[3]. Singh[4] extended some elementary properties of convex sets and functions to arcwise connected sets and functions. Yadav and Mukherjee[5] studied some basic properties for a class of generalized arcwise connected functions.

In section 2 of this paper, we define arcwise combinations of more than two points and use it to characterize arcwise connected sets. We introduce the concepts of strictly arcwise connected functions, semistrictly arcwise connected functions and arcwise affine functions as generalizations of strictly convex functions, semistrictly convex functions and affine functions respectively. In Section 3, we extend the Jensen's inequality[6] for convex and strictly convex functions to arcwise connected and strictly arcwise connected functions. Jensen's inequality was generalized for semistrictly convex function by Yang[7]. We extend this result for semistrictly arcwise connected function. In Section 4 of this paper, we establish the relationships among arcwise connected functions, strictly arcwise connected functions and semistrictly arcwise connected functions. It is shown that the class of arcwise connected functions is independent of the class of semistrictly arcwise whereas the class of strictly arcwise connected functions is contained in the classes of both arcwise connected and semistrictly arcwise connected functions. We discuss the conditions under which an arcwise connected function is strictly arcwise connected or semistrictly arcwise connected. We also show that under certain assumptions a semistrictly arcwise connected function is arcwise connected or strictly arcwise connected.

2. Preliminaries

The following definitions of arcwise connected set and arcwise connected functions are by Avriel and Zang[3].

Definition 2.1 — A subset X of R^m is said to be an arcwise connected set if, for any pair of points $x, y \in X$ there exists a continuous arc $H_{x,y}(\alpha)$ defined on [0, 1] with a value in X such that $H_{x,y}(0) = x$ and $H_{x,y}(1) = y$.

Definition 2.2 — A function $f: X \to R$, where X is an arcwise connected set, is said to be an arcwise connected function if, for any $x, y \in X$, $\alpha \in [0, 1]$

$$f(H_{x,y}(\alpha)) \leq (1-\alpha) f(x) + \alpha f(y).$$

Note that a convex set is arcwise connected set where the function $H_{x,y}(\alpha)$ is given by

$$H_{x,y}(\alpha) = (1-\alpha)\, x + \alpha\, y$$

for any $x, y \in X$, $\alpha \in [0, 1]$. Similarly, a convex function is an arcwise connected function.

We now extend the concept of convex combination of more than two points to arcwise combination.

Definition 2.3 — Let $x_1, x_2, \ldots, x_n \in R^m$. Then

$D_{x_1, x_2, \ldots, x_n}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is said to be an arcwise combination of $x_1, x_2, \ldots, x_n$ where

$$D_{x_1, x_2, \ldots, x_n}(\alpha_1, \alpha_2, \ldots, \alpha_n) = H_{y_{n-1}, x_n}(\alpha_n),$$

$$y_{n-1} = H_{y_{n-2}, x_{n-1}}\left(\frac{\alpha_{n-1}}{\alpha_1 + \alpha_2 + \ldots + \alpha_{n-1}}\right),$$

$$y_{n-2} = H_{y_{n-3}, x_{n-2}}\left(\frac{\alpha_{n-2}}{\alpha_1 + \alpha_2 + \ldots + \alpha_{n-2}}\right),$$

....

$$y_2 = H_{y_1, x_2}\left(\frac{\alpha_2}{\alpha_1 + \alpha_2}\right),$$

$$y_1 = x_1$$

and

$$\alpha_i \geq 0,\ i = 1, 2, \ldots, n,\ \sum_{i=1}^{n} \alpha_i = 1, n \geq 2.$$

The next theorem characterizes arcwise connectedness of a set in terms of arcwise combinations.

Theorem 2.1 — *A subset X of R^m is arcwise connected, if and only if, it contains all the arcwise combinations of its elements.*

PROOF : A subset X of R^m is arcwise connected if and only if, for any $x_1, x_2 \in X, H_{x_1, x_2}(0) = x_1$, $H_{x_1, x_2}(1) = x_2$ and $H_{x_1, x_2}(\alpha) \in X$, for $\alpha \in [0, 1]$. If X is arcwise connected then it contains arcwise combinations of two elements, that is, the result is true for $n = 2$. For $n > 2$ we will prove the result by mathematical induction. Let the result be true for all arcwise combinations of fewer than n elements. Consider an arcwise combination of $x_1, x_2, \ldots, x_n$ given by $D_{x_1, x_2, \ldots, x_n}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ where $\alpha_i \geq 0$, $i = 1, 2, .., n$ and $\sum_{i=1}^{n} \alpha_i = 1$. Since $\sum_{i=1}^{n} \alpha_i = 1$, at least one $\alpha_i \neq 1$. Without loss of generality, assume $\alpha_n \neq 1$. Now

$$D_{x_1, x_2, \ldots, x_n}(\alpha_1, \alpha_2, \ldots, \alpha_n) = H_{y_{n-1}, x_n}(\alpha_n),$$

where

$$y_{n-1} = H_{y_{n-2}, x_{n-1}}\left(\frac{\alpha_{n-1}}{\alpha_1 + \alpha_2 + \ldots + \alpha_{n-1}}\right) = H_{y_{n-2}, x_{n-1}}\left(\frac{\alpha_{n-1}}{1 - \alpha_n}\right)$$

$$= D_{x_1, x_2, \ldots, x_{n-1}}\left(\frac{\alpha_1}{1-\alpha_n}, \frac{\alpha_2}{1-\alpha_n}, \ldots, \frac{\alpha_{m-1}}{1-\alpha_n}\right)$$

and $\frac{\alpha_i}{1-\alpha_n} \geq 0$, $i = 1, 2, \ldots, n-1$, $\sum_{i=1}^{n-1} \frac{\alpha_i}{1-\alpha_n} = 1$. Now by induction hypothesis $y_{n-1} \in X$. As X is an arcwise connected set and $y_{n-1}, x_n \in X$ it follows that $H_{y_{n-1}, x_n}(\alpha_n) \in X$, that is, $D_{x_1, x_2, \ldots, x_n}(\alpha_1, \alpha_2, \ldots, \alpha_n) \in X$.

We now define strict arcwise connectedness, semistrict arcwise connectedness and arcwise affinity.

Definition 2.4 — A function $f: X \to R$, where X is an arcwise connected set, is said to be

a) strictly arcwise connected, if for any $x, y \in X, x \neq y$, $\alpha \in (0, 1)$

$$f(H_{x,y}(\alpha)) < (1-\alpha) f(x) + \alpha f(y),$$

b) semistrictly arcwise connected, if for any $x, y \in X$, $f(x) \neq f(y)$, $\alpha \in (0, 1)$

$$f(H_{x,y}(\alpha)) < (1-\alpha) f(x) + \alpha f(y)$$

c) arcwise affine, if for any $x, y \in X$, $\alpha \in [0, 1]$

$$f(H_{x,y}(\alpha)) = (1-\alpha) f(x) + \alpha f(y).$$

We will now establish that strict arcwise connectedness, semistrict arcwise connectedness, arcwise affinity are generalizations of strict convexity, semistrict convexity [Yang[7]], affinity, respectively.

Remark 2.1 : Every strictly convex function is strictly arcwise connected but the converse is not necessarily true as illustrated by the following example.

Example 2.1 — Let $X = R$. Define $f: X \to R$ by $f(x) = |x|$.

For any $x, y \in X$, define $H_{x,y}(\alpha): [0, 1] \to R$ by

$$H_{x,y}(\alpha) = (1-\alpha)^2 x + \alpha^2 y.$$

Clearly, f is strictly arcwise connected but not strictly convex because for $x = 1$, $y = 2$, $\alpha = \frac{1}{2}$

$$f((1-\alpha)x + \alpha y) \not< (1-\alpha)f(x) + \alpha f(y).$$

Remark 2.2 : Every semistrictly convex function is semistrictly arcwise connected but the converse is not necessarily true as illustrated by the following example.

Example 2.2 — Let $X = R$. Define $f: X \to R$ by

$$f(x) = \begin{cases} 0 & x < 0 \\ x & x \geq 0. \end{cases}$$

For any $x, y \in X$, define $H_{x,y}(\alpha): [0, 1] \to R$ by

$$H_{x,y}(\alpha) = (1-\alpha)^2 x + \alpha^2 y.$$

It can be seen that f is semistrictly arcwise connected but not semistrictly convex because for $x = 1$ $y = 3$, $\alpha = \frac{1}{2}$, $f(x) \neq f(y)$.

$$f((1-\alpha)x + \alpha y) \not< (1-\alpha)f(x) + \alpha f(y).$$

Remark 2.3 : Every affine function is arcwise affine. However the converse is not necessarily true as illustrated by the following example.

Example 2.3 — Let $X = R_+ \setminus \{0\}$. Define $f: X \to R$ by $f(x) = \log x$. For any $x, y \in X$, define $H_{x,y}(\alpha): [0, 1] \to R$ by

$$H_{x,y}(\alpha) = x^{1-\alpha} y^{\alpha}.$$

Clearly, $H_{x,y}(0) = x$, $H_{x,y}(1) = y$ and $H_{x,y}(\alpha) \in X$ for $\alpha \in [0, 1]$. It can be seen that f is arcwise affine but not affine because for $x = 1$, $y = 2$, $\alpha = \frac{1}{2}$

$$f((1-\alpha)x + \alpha y) \neq (1-\alpha)f(x) + \alpha f(y).$$

3. Extensions of Jensen's Inequality

Jensen's[6] inequality for convex function and strictly convex function is stated below :

Theorem 3.1 — *Let $f: X \to R$ be a function defined on a convex set X in R^m. Then f is convex, if and only if,*

$$f(\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2) + \ldots + \alpha_n f(x_n) \qquad \ldots (1)$$

for every integer $n \geq 2$ and all $x_i \in X$, $\alpha_i \geq 0$, $i = 1, 2, ..., n$ with $\sum_{i=1}^{n} \alpha_i = 1$. If f is strictly convex and all $\alpha_i > 0$ then equality occurs in (1), if and only if, $x_1 = x_2 = \ldots = x_n$.

We now extend the above result for arcwise connected and strictly arcwise connected functions.

Theorem 3.2 — *Let $f: X \to R$ be a function defined on an arcwise connected set X in R^m. Then f is arcwise connected if and only if*

$$f(D_{x_1, x_2, \ldots, x_n}(\alpha_1, \alpha_2, \ldots, \alpha_n)) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2) + \ldots + \alpha_n f(x_n)$$

for every integer $n \geq 2$ and all $x_i \in X$, $\alpha_i \geq 0$, $i = 1, 2, ..., n$ with $\sum_{i=1}^{n} \alpha_i = 1$.

Proof : Let f be an arcwise connected function. Then

$$f(y_2) = f\left(H_{y_1, x_2}\left(\frac{\alpha_2}{\alpha_1 + \alpha_2}\right)\right) \leq \left(1 - \frac{\alpha_2}{\alpha_1 + \alpha_2}\right) f(y_1) + \frac{\alpha_2}{\alpha_1 + \alpha_2} f(x_2)$$

$$= \frac{1}{\alpha_1 + \alpha_2} \sum_{i=1}^{2} \alpha_i f(x_i) \qquad \ldots (2)$$

and

$$f(y_3) = f\left(H_{y_2, x_3}\left(\frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3}\right)\right) \leq \frac{1}{\alpha_1 + \alpha_2 + \alpha_3} [(\alpha_1 + \alpha_2) f(y_2) + \alpha_3 f(x_3)]. \qquad \ldots (3)$$

On using (2) in (3), we get

$$f(y_3) \leq \frac{1}{\alpha_1 + \alpha_2 + \alpha_3} \sum_{i=1}^{3} \alpha_i f(x_i).$$

Proceeding similarly, we get

$$f(y_{n-1}) \leq \frac{1}{\alpha_1 + \alpha_2 + \ldots + \alpha_{n-1}} \sum_{i=1}^{n-1} \alpha_i f(x_i) \qquad \ldots (4)$$

and

$$f(D_{x_1, x_2, \ldots, x_n}(\alpha_1, \alpha_2, \ldots, \alpha_n)) = f(H_{y_{n-1}, x_n}(\alpha_n))$$

$$\leq (1 - \alpha_n) f(y_{n-1}) + \alpha_n f(x_n). \qquad \ldots (5)$$

Since $1 - \alpha_n = \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1}$, from (4) and (5), we get

$$f(D_{x_1, x_2, \ldots, x_n}(\alpha_1, \alpha_2, \ldots, \alpha_n)) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2) + \ldots + \alpha_n f(x_n).$$

Converse is obvious by taking $n = 2$.

Theorem 3.3 — *Let $f: X \to R$ be a strictly arcwise connected function defined on an arcwise connected set X in R^m. We assume that $H_{x,x}(\alpha) = x$ for $\alpha \in [0, 1]$. Then for every integer $n \geq 2$ and all $x_i \in X$, $\alpha_i > 0$, $i = 1, 2, \ldots, n$ with $\sum_{i=1}^{n} \alpha_i = 1$, we have*

$$f(D_{x_1, x_2, \ldots, x_n}(\alpha_1, \alpha_2, \ldots, \alpha_n)) = \alpha_1 f(x_1) + \alpha_2 f(x_2) + \ldots + \alpha_n f(x_n) \qquad \ldots (6)$$

if and only if $x_1 = x_2 = \ldots = x_n$.

PROOF : For $x_i \in X$, $\alpha_i > 0$, $i = 1, 2, \ldots, n$ with $\sum_{i=1}^{n} \alpha_i = 1$, let relation (6) hold. Let if possible, $x_1 \neq x_2$. As f is strictly arcwise connected function, we have

$$f(y_2) < \frac{1}{\alpha_1 + \alpha_2} \sum_{i=1}^{2} \alpha_i f(x_i). \qquad \ldots (7)$$

Either $y_2 = x_3$ or $y_2 \neq x_3$. If $y_2 = x_3$ then

$$f(y_3) = f\left(H_{y_2, x_3}\left(\frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3}\right)\right)$$

$$= f(y_2)$$

$$= f(x_3)$$

$$= \frac{1}{\alpha_1 + \alpha_2 + \alpha_3}[(\alpha_1 + \alpha_2) f(y_2) + \alpha_3 f(x_3)].$$

If $y_2 \neq x_3$, by strict arcwise connectedness of f, we get

$$f(y_3) < \frac{1}{\alpha_1 + \alpha_2 + \alpha_3}[(\alpha_1 + \alpha_2) f(y_2) + \alpha_3 f(x_3)].$$

In either case using (7), we get

$$f(y_3) < \frac{1}{\alpha_1 + \alpha_2 + \alpha_3} \sum_{i=1}^{3} \alpha_i f(x_i).$$

Proceeding similarly, we get

$$f(D_{x_1, x_2, \dots, x_n}(\alpha_1, \alpha_2, \dots, \alpha_n)) < \alpha_1 f(x_1) + \alpha_2 f(x_2) + \dots + \alpha_n f(x_n) \tag{8}$$

which contradicts (6). Hence $x_1 = x_2$. By definition of y_1 and y_2, we get

$$y_1 = y_2 = x_1 = x_2$$

and hence

$$f(y_2) = f\left(H_{y_1, x_2}\left(\frac{\alpha_2}{\alpha_1 + \alpha_2}\right)\right) = \frac{1}{\alpha_1 + \alpha_2} \sum_{i=1}^{2} \alpha_i f(x_i). \tag{9}$$

Now if $x_2 \neq x_3$, we have $y_2 \neq x_3$. As f is strictly arcwise connected function it follows that

$$f(y_3) < \frac{1}{\alpha_1 + \alpha_2 + \alpha_3} [(\alpha_1 + \alpha_2) f(y_2) + \alpha_3 f(x_3)].$$

On using (9), we get

$$f(y_3) < \frac{1}{\alpha_1 + \alpha_2 + \alpha_3} \sum_{i=1}^{3} \alpha_i f(x_i).$$

Proceeding similarly we arrive at (8) which contradicts (6).
Hence, $x_2 = x_3$. In a similar manner it can be proved that $x_3 = x_4, x_4 = x_5, \dots, x_{n-1} = x_n$.
Conversely, let $x_1 = x_2 = \dots = x_n$. Then it follows that —

$$y_1 = x_1$$

$$y_2 = y_1 = x_2$$

$$y_3 = y_2 = x_3$$

....

$$y_{n-1} = y_{n-2} = x_{n-1}$$

$$D_{x_1, x_2, \dots, x_n}(\alpha_1, \alpha_2, \dots, \alpha_n) = y_{n-1} = x_n.$$

Therefore, we get

$$D_{x_1, x_2, \ldots, x_n}(\alpha_1, \alpha_2, \ldots, \alpha_n) = x_1 = x_2 = \ldots. = x_n$$

and hence for $\alpha_i > 0$, $i = 1, 2, \ldots, n$ with $\sum_{i=1}^{n} \alpha_i = 1$, we have

$$f(D_{x_1, x_2, \ldots, x_n}(\alpha_1, \alpha_2, \ldots, \alpha_n)) = (\alpha_1 + \alpha_2 + \ldots + \alpha_n) f(x_1)$$

$$= \alpha_1 f(x_1) + \alpha_2 f(x_2) + \ldots + \alpha_n f(x_n).$$

The following theorem generalizes Theorem 1 of Yang[7] which was given for semistrictly convex function.

Theorem 3.4 — *Let $f: X \to R$ be a function defined on an arcwise connected set X in R^m. Then f is semistrictly arcwise connected, if and only if, for every integer $n \geq 2$ and all $x_i \in X$, $\alpha_i > 0$ $i = 1, 2, \ldots, n$ with $\sum_{i=1}^{n} \alpha_i = 1, f(x_1) < f(x_2) \leq f(x_3) \leq \ldots \leq f(x_n)$, we have*

$$f(D_{x_1, x_2, \ldots, x_n}(\alpha_1, \alpha_2, \ldots, \alpha_n)) < \alpha_1 f(x_1) + \alpha_2 f(x_2) + \ldots + \alpha_n f(x_n).$$

PROOF : Let f be a semistrictly arcwise connected function. For every integer $n \geq 2$, all $x_i \in X$, $\alpha_i > 0$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} \alpha_i = 1$, let $f(x_1) < f(x_2) \leq f(x_3) \leq \ldots \leq f(x_n)$. As f is semistrictly arcwise connected, we have

$$f(y_2) < \frac{1}{\alpha_1 + \alpha_2} \sum_{i=1}^{2} f(x_i) < f(x_2) \leq f(x_3).$$

As $f(y_2) \neq f(y_3)$, by semistrict arcwise connectedness, it follows that

$$f(y_3) < \frac{1}{\alpha_1 + \alpha_2 + \alpha_3} [(\alpha_1 + \alpha_2) f(y_2) + \alpha_3 f(x_3)]$$

$$< \frac{1}{\alpha_1 + \alpha_2 + \alpha_3} \sum_{i=1}^{3} \alpha_i f(x_i) < f(x_3) \leq f(x_4).$$

Proceeding similarly, we get

$$f(y_{n-1}) < \frac{1}{\alpha_1 + \alpha_2 + \ldots + \alpha_{n-1}} \sum_{i=1}^{n-1} \alpha_i f(x_i) < f(x_{n-1}) \leq f(x_n).$$

Hence

$$f(D_{x_1, x_2, \ldots, x_n}(\alpha_1, \alpha_2, \ldots, \alpha_n)) < \left(\sum_{i=1}^{n-1} \alpha_i \right) f(y_{n-1}) + \alpha_n f(x_n)$$

$$< \alpha_1 f(x_1) + \alpha_2 f(x_2) + \ldots + \alpha_n f(x_n).$$

Converse is obvious by taking $n = 2$.

4. Relationships Among Arcwise Connected and Related Functions

We first discuss the relation between an arcwise connected and a strictly arcwise connected function.

Remark 4.1 : Every strictly arcwise connected function is arcwise connected. However the converse is not necessarily true. The function f considered in Example 2.3 is arcwise affine and hence arcwise connected but is not strictly arcwise connected.

A function $f: X \to R$ defined on an arcwise connected set X in R^n is said to be arcwise nonaffine if for $x, y \in X$,

$$F(H_{x,y}(\alpha)) \neq (1-\alpha) f(x) + \alpha f(y)$$

for some $\alpha \in (0, 1)$.

Theorem 4.1 below illustrates that an arcwise connected function is strictly arcwise connected if it is arcwise nonaffine.

Theorem 4.1 — *Let $f: X \to R$ be an arcwise connected function defined on an arcwise connected set X in R^n :-*

a) If f is arcwise nonaffine then for any $x, y \in X$, $\alpha \in (0, 1)$

$$f(H_{x,y}(\alpha)) < (1-\alpha) f(x) + \alpha f(y).$$

b) If there exists $\alpha \in (0, 1)$ such that for any $x, y \in X$, $x \neq y$ implies

$$f(H_{x,y}(\alpha)) < (1-\alpha) f(x) + \alpha f(y)$$

then f is a strictly arcwise connected function.

Proof : *a)* Since f is an arcwise connected function then for $x, y \in X$, we have

$$F(\alpha) \equiv (1-\alpha) f(x) + \alpha f(y) - f(H_{x,y}(\alpha)) \geq 0 \quad \ldots (10)$$

for all $\alpha \in (0, 1)$. We have to prove that $F(\alpha) > 0$ for all $\alpha \in (0, 1)$. Assume to the contrary that there exists $\alpha_0 \in (0, 1)$ such that $F(\alpha_0) = 0$. Let $\alpha \in (0, \alpha_0)$. Let $z_0 = H_{x,y}(\alpha_0)$ and $z = H_{x,y}(\alpha)$. Define

$$H_{z,y}(\theta) = H_{x,y}(\alpha + (1-\alpha)\theta).$$

Then

$$z_0 = H_{z,y}\left(\frac{\alpha_0 - \alpha}{1-\alpha}\right),$$

where $\frac{\alpha_0 - \alpha}{1-\alpha} \in (0, 1)$. Since f is arcwise connected and $F(\alpha_0) = 0$ we have

$$(1-\alpha_0) f(x) + \alpha_0 f(y) = f(z_0) \leq \left(1 - \frac{\alpha_0 - \alpha}{1-\alpha}\right) f(z) + \frac{(\alpha_0 - \alpha)}{1-\alpha} f(y)$$

which on simplification gives

$$(1-\alpha) f(x) + \alpha f(y) \leq f(z)$$

that is, for $\alpha \in (0, \alpha_0)$

$$F(\alpha) \equiv (1-\alpha) f(x) + \alpha f(y) - f(H_{x,y}(\alpha)) \leq 0. \quad \ldots (11)$$

Let $\alpha \in (\alpha_0, 1)$. Define

$$H_{x,z}(\theta) = H_{x,y}(\alpha\, \theta)$$

Then

$$z_0 = H_{x,z}\left(\frac{\alpha_0}{\alpha}\right),$$

where $\frac{\alpha_0}{\alpha} \in (0, 1)$. Since f is arcwise connected and $F(\alpha_0) = 0$ we have

$$(1-\alpha_0) f(x) + \alpha_0 f(y) = f(z_0) \leq \left(1 - \frac{\alpha_0}{\alpha}\right) f(x) + \frac{\alpha_0}{\alpha} f(z)$$

that is, for $\alpha \in (\alpha_0, 1)$.

$$F(\alpha) \equiv (1-\alpha) f(x) + \alpha f(y) - f(H_{x,y}(\alpha)) \leq 0. \quad \ldots (12)$$

From (10), (11) and (12) it follows that $F(\alpha) = 0$ for all $\alpha \in (0, 1)$ which is a contradiction as f is an arcwise nonaffine function.

b) Since there exists an $\alpha \in (0, 1)$ such that for $x, y \in X, x \neq y$

$$f(H_{x,y}(\alpha)) < (1-\alpha) f(x) + \alpha f(y)$$

it follows that f is arcwise nonaffine. The proof in this case therefore follows from *a*).

Next, we discuss the relations between arcwise connected and semistrictly arcwise connected functions.

Remark 4.2 : An arcwise connected function may not be semistrictly arcwise connected. The function f considered in Example 2.3 is arcwise connected but not semistrictly arcwise connected.

The following theorem gives the conditions under which an arcwise connected function is semistrictly arcwise connected.

Theorem 4.2 — *Let* $f: X \rightarrow R$ *be an arcwise connected function defined on an arcwise connected set* X *in* R^n. *If there exists* $\alpha \in (0, 1)$, *such that, for any* $x, y \in X, f(x) \neq f(y)$ *implies*

$$f(H_{x,y}(\alpha)) < (1-\alpha) f(x) + \alpha f(y)$$

then f is a semistrictly arcwise connected function.

PROOF : Proof follows on the lines of Theorem 4.1 (*b*).

Remark 4.3 : A semistrictly arcwise connected function may not be arcwise connected which is illustrated by the following example.

Example 4.1 — Let $X = R^+ \setminus \{0\}$. Define $f: X \rightarrow R$ by

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases}.$$

Define $H_{x,y}(\alpha) : [0, 1] \rightarrow R$ by

$$H_{x,y}(\alpha) = (1-\alpha)^2 x + \alpha^2 y.$$

Clearly f is semistrictly arcwise connected but not arcwise connected because for

$$x = \frac{1}{3},\ y = 13,\ \alpha = \frac{1}{4}$$

$$f(H_{x,y}(\alpha)) \nleq (1-\alpha) f(x) + \alpha f(y).$$

The following theorem can be established on the lines of the corresponding result by Yang[7] on the lines of work by Mehra and Bhatia[8]. This theorem gives the condition under which a semistrictly arcwise connected function is arcwise connected.

Theorem 4.3 — *Let* $f: X \rightarrow R$ *be a semistrictly arcwise connected function defined on an arcwise connected set* X *in* R^n. *If there exists* $\alpha \in (0, 1)$ *such that, for any* $x, y \in X$

$$f(H_{x,y}(\alpha)) \leq (1-\alpha) f(x) + \alpha f(y)$$

then f *is an arcwise connected function.*

We now discuss the relationship between strictly arcwise connected and semistrictly arcwise connected functions.

Remark 4.4 : Every strictly arcwise connected function is semistrictly arcwise connected. However, the converse is not necessarily true. The function f considered in Example 2.2 is semistrictly arcwise connected but not strictly arcwise connected because for $x = -2$, $y = 0$, $\alpha = \frac{1}{2}$

$$f(H_{x,y}(\alpha)) \nless (1-\alpha) f(x) + \alpha f(y).$$

In the following theorem, we state the condition under which a semistrictly arcwise connected function is strictly arcwise connected.

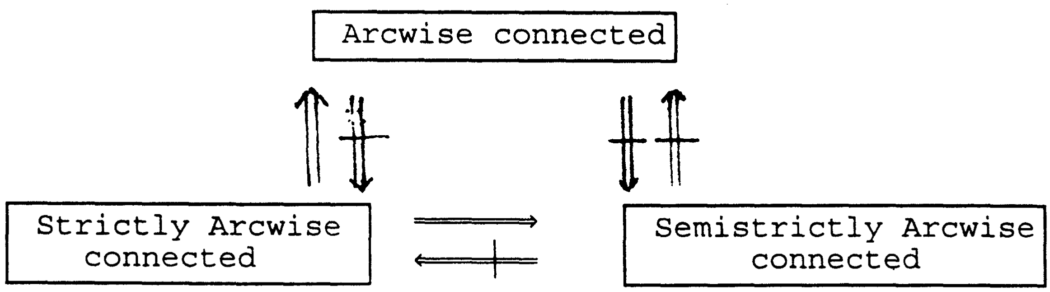
Theorem 4.4 — *Let $f : X \to R$ be a semistrictly arcwise connected function defined on an arcwise connected set X in R^n. If there exists $\alpha \in (0, 1)$ such that for any $x, y \in X, x \neq y$ implies*

$$f(H_{x,y}(\alpha)) < (1-\alpha) f(x) + \alpha f(y)$$

then f is a strictly arcwise connected function.

PROOF : Proof follows on the lines of Yang[7] and Mehra and Bhatia[8].

We now summarize the relationships among the above discussed functions as follows



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