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AN EXPLICIT FORMULA ON THE GENERALIZED BERNOULLI NUMBER WITH ORDER n

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The purpose of this paper is to define the extension number of Bernoulli number with order n given by Srivastava and Todorov[2] and to prove a new explicit formula for this number.

Key Words : Bernoulli Number; Zeta Function

INTRODUCTION

For any complex number x, it is well known that the Bernoulli polynomial $B_n(x)$ is defined by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n, \quad |z| < 2\pi.$$

Note that $B_n(0)$ is the nth Bernoulli number.

In [2], the Bernoulli polynomials and numbers with order n are defined by

$$\left(\frac{z}{e^z - 1}\right)^n e^{xz} = \sum_{m=0}^{\infty} B_m^{(n)}(x) \frac{z^m}{m!}$$

and

$$\left(\frac{z}{e^z - 1}\right)^n = \sum_{m=0}^{\infty} B_m^{(n)} \frac{z^m}{m!}, \quad |z| < 2\pi.$$

It is easy to see that

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$$B_m^{(n)}(x) = (-1)^m B_m^{(n)}(n-x)$$

and

$$B_m^{(n)}(n) = (-1)^m B_m^{(n)}(0) \equiv (-1)^m B_m^{(n)} \qquad \dots (1)$$

in terms of the generalized Bernoulli numbers $B_m^{(n)}$[2].

To define a generalized Bernoulli number with order n, we first mention the multiple zeta values and multiple logarithm,

Don Zagier devoted himself to the study of the multiple zeta values, and mentioned briefly the multiple zeta function as follows :

$$\zeta(s_1, s_2, \dots, s_n) = \sum_{0 < m_1 < m_2 < \dots , m_n} \frac{1}{m_1^{s_1} m_2^{s_2} \dots m_n^{s_n}}, \; s_i \in \mathbb{C}.$$

In particular, it is analytically continued to a meromorphic function on $\mathbb{C}^n$.

Now, we study the following zeta function :

$$\zeta(k_1, k_2, \dots, k_{n-1}; s) = \sum_{0 < m_1 < m_2 < \dots < m_n}^{\bullet} \frac{1}{m_1^{k_1} m_2^{k_2} \dots m_{n-1}^{k_{n-1}} m_n^{s}}$$

for $n \geq 1, k_i \geq 1 \; (1 \leq i \leq n-1)$, which converges absolutely for $Re(s) > 1$.

It is already known that a multiple logarithm is defined by

$$Li_{k_1, k_2, \dots, k_n}(z) = \sum_{0 < m_1 < m_2 < \dots < m_n} \frac{z^{m_n}}{m_1^{k_1} m_2^{k_2} \dots m_n^{k_n}},$$

for $k_i \geq 1 \; (1 \leq i \leq n), |z| < 1$, the multiple logarithmic function is closely related to multiple zeta values, i.e.,

$$Li_{k_1, k_2, \dots, k_n}(1) = \zeta(k_1, k_2, \dots, k_n), \; k_i \geq 1 \;\; (1 \leq i \leq n-1), \; k_n \geq 2.$$

The multi-logarithm naturally appears in the theory of knot invariants.

In this paper, the aim is to define the extension number of Bernoulli number with order n given by Srivastava and Todorov[2]. We prove a new explicit formula for this number.

§ 1. Generalized Bernoulli Numbers with Order n

For $k_i \geq 1 \; (1 \leq i \leq n-1)$ and $k_n = 1$, it is easy to see that

$$\frac{d}{dz} Li_{k_1, k_2, \dots, k_{n-1}, 1}(z) = \frac{1}{1-z} Li_{k_1, k_2, \dots, k_{n-1}}(z).$$

From this, we have

$$\frac{d}{dz} Li_{k,1}(z) = \frac{1}{1-z} Li_k(z).$$

So if $k = 1$, then we have

$$\frac{d}{dz} Li_{1,1}(z) = \frac{1}{1-z} Li_1(z) = -\frac{1}{1-z} \log(1-z),$$

where the poly-logarithm $Li_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}, |z| < 1.$

Hence,

$$Li_{1,1}(z) = \int_0^z \frac{d}{dz} Li_{1,1}(z)\, dz = \int_0^z \frac{1}{1-z}(-\log(1-z))\, dz = \frac{1}{2!}(-\log(1-z))^2.$$

Also, from the above differentiation, we obtain

$$\frac{d}{dz} Li_{1,1,1}(z) = \frac{1}{1-z} Li_{1,1}(z).$$

Hence,

$$Li_{1,1,1}(z) = \int_0^z \frac{d}{dz} Li_{1,1,1}(z)\, dz = \frac{1}{3!}(-\log(1-z))^3.$$

Continuing this process, we have

$$Li_{\underbrace{1,1,\ldots,1}_{n-times}}(z) = \frac{1}{n!}(-\log(1-z))^n. \qquad \ldots (2)$$

For each integer k_i $(i = 1, 2, \ldots, n)$, we define a sequence of rational numbers $B_m^{(k_1, k_2, \ldots, k_n)}$ $(m = 0, 1, 2, \ldots)$ as generalized Bernoulli numbers with order n by

$$\left.\frac{Li_{k_1, k_2, \ldots, k_n}(z)}{z^n}\right|_{z = 1 - e^{-x}} = \sum_{m=0}^{\infty} \frac{B_m^{(k_1, k_2, \ldots, k_n)}}{m!} x^m.$$

Thus

$$\left.\frac{Li_{\underbrace{1,1,1}_{n-times}}(z)}{z^n}\right|_{z = 1 - e^{-x}} = \sum_{m=0}^{\infty} \frac{B_m^{(\overbrace{1,1,\ldots,1}^{n-times})}}{m!} x^m. \qquad \ldots (3)$$

From the eq. (2), the left hand side of (2) is written by

$$\left.\frac{Li_{\underbrace{1, 1, \ldots, 1}_{n-times}}(z)}{z^n}\right|_{z=1-e^{-x}} = \frac{1}{n!}\left(\frac{x}{1-e^{-x}}\right)^n$$

$$= \frac{1}{n!}\left(\frac{-x}{e^{-x}-1}\right)^n$$

$$= \frac{1}{n!}\sum_{m=0}^{\infty}\frac{B_m^{(n)}(-1)^m}{m!}x^m.$$

Note that

$$B_m^{(\overbrace{1, 1, \ldots, 1}^{n-times})} = \frac{(-1)^m B_m^{(n)}}{n!} = \frac{B_m^{(n)}(n)}{n!},$$

where $B_m^{(n)}$ is the mth Bernoulli number with order n which is found in [2].

The Stirling numbers of the second kind $S(n, m)$ $(n \geq 0, 0 \leq m < n)$ are defined by

$$x^n = \sum_{m=0}^{n} S(n, m)(x)_m,$$

in which

$$S(n, m) = \frac{(-1)^m}{m!}\sum_{l=0}^{m}(-1)^l\binom{m}{l}l^n$$

and

$$\frac{(e^t-1)^m}{m!} = \sum_{n=m}^{\infty} S(n, m)\frac{t^n}{n!}, \qquad \ldots (4)$$

where $(x)_m$ is denoted by Jordan factor.

For $n > 1$, by the definition of a generating function of $B_m^{(k_1, k_2, \ldots, k_n)}$ and by the formula (4), we have

$$\left.\frac{Li_{k_1, k_2, \ldots, k_{n-1}, -k_n}(z)}{z^n}\right|_{z=1-e^{-x}}$$

$$= \frac{1}{(1-e^{-x})^n}\sum_{0<m_1<\ldots<m_n}\frac{(1-e^{-x})^{m_n}}{m_1^{k_1}m_2^{k_2}\ldots m_{n-1}^{k_{n-1}}m_n^{-k_n}}$$

$$= \frac{1}{(1-e^{-x})^n} \sum_{0<m_1<\dots<m_{n-1}} \frac{1}{m_1^{k_1} m_2^{k_2} \dots m_{n-1}^{k_{n-1}}} \sum_{m_n=m_{n-1}+1}^{\infty} \frac{(1-e^{-x})^{m_n}}{m_n^{-k_n}}$$

$$= \frac{1}{(1-e^{-x})^{n-1}} \sum_{0<m_1<\dots<m_{n-1}} \frac{(1-e^{-x})^{m_{n-1}}}{m_1^{k_1} m_2^{k_2} \dots m_{n-1}^{k_{n-1}}}$$

$$\times \sum_{l=0}^{\infty} \sum_{m_n=0}^{l} \frac{(-1)^{m_n+1} m_n!\, S(l, m_n)}{(m_n+m_{n-1}+1)^{-k_n}} \frac{x^l}{l!}$$

$$= \sum_{l=0}^{\infty} \left(\sum_{m_n=0}^{l} \sum_{k_n=a_1+a_2+a_3} (-1)^{m_n+1} m_n!\, S(l, m_n) \frac{(a_1+a_2+a_3)!}{a_1!\, a_2!\, a_3!} m_n^{a_2} \times \frac{Li_{k_1, k_2, \dots, k_{n-1}-a_1}(1-e^{-x})}{(1-e^{-x})^{n-1}} \frac{x^l}{l!} \right)$$

$$= \sum_{p=0}^{\infty} \left(\sum_{l=0}^{p} \frac{1}{\binom{p}{l}} \sum_{m_n=0}^{l} \sum_{k_n=a_1+a_2+a_3} (-1)^{m_n+l} m_n!\, S(l, m_n) \frac{(a_1+a_2+a_3)!}{a_1!\, a_2!\, a_3!} m_n^{a_2} \times B_{p-l}^{(k_1, k_2, \dots, k_{n-1}-a_1)} \right) \frac{x^p}{p!}.$$

Therefore, we obtain the formula :

$$B_p^{(k_1, k_2, \dots, k_{n-1}, -k_n)}$$

$$= \sum_{l=0}^{p} \frac{1}{\binom{p}{l}} \sum_{m_n=0}^{l} (-1)^{m_n+l} m_n!\, S(l, m_n) \sum_{k_n=a_1+a_2+a_3} \frac{(a_1+a_2+a_3)!}{a_1!\, a_2!\, a_3!} m_n^{a_2}$$

$$\times B_{p-l}^{(k_1, k_2, \dots, k_{n-1}-a_1)}.$$

2. Relations Between Multiple Zeta Values and Generalized Bernoulli Numbers with Order n

Now we redefine the generalized Bernoulli numbers with order n as :

$$\frac{Li_{k_1, k_2, \ldots, k_n}(1-e^{-t})}{(e^t-1)^n} = \sum_{m=0}^{\infty} \frac{B_m^{(k_1, k_2, \ldots, k_n)}}{m!} t^m$$

for $n = 1, 2, 3, \ldots$.

Consider the generalized multiple zeta function $\zeta_n(k_1, k_2, \ldots, k_n; s)$ as

$$\zeta_n(k_1, k_2, \ldots, k_n; s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{(e^t-1)^n} Li_{k_1, k_2, \ldots, k_n}(1-e^{-t})\, dt.$$

Thus we see that

$$\zeta_n(k_1, k_2, \ldots, k_n; s) = \frac{1}{\Gamma(s)} \int_0^{1} \frac{t^{s-1}}{(e^t-1)^n} Li_{k_1, k_2, \ldots, k_n}(1-e^{-t})\, dt$$

$$+ \frac{1}{\Gamma(s)} \int_1^{\infty} \frac{t^{s-1}}{(e^t-1)^n} Li_{k_1, k_2, \ldots, k_n}(1-e^{-t})\, dt.$$

The second integral converges absolutely for any $s \in \mathbb{C}$ and hence the second term on the right hand side vanishes at non-positive integers. Hence, we have

$$\zeta_n(k_1, k_2, \ldots, k_n; -m) = \lim_{s \to -m} \frac{1}{\Gamma(s)} \int_0^{1} \frac{t^{s-1}}{(e^t-1)^n} Li_{k_1, k_2, \ldots, k_n}(1-e^{-t})\, dt$$

$$= \lim_{s \to -m} \frac{1}{\Gamma(s)} \int_0^{1} t^{s-1} \sum_{l=0}^{\infty} \frac{B_l^{(k_1, k_2, \ldots, k_n)}}{l!} t^l\, dt$$

$$= \lim_{s \to -m} \frac{1}{\Gamma(s)} \sum_{l=0}^{\infty} \frac{B_l^{(k_1, k_2, \ldots, k_n)}}{l!} \frac{1}{l+s}$$

$$= \lim_{s \to -m} \frac{\Gamma(1-s)\sin \pi s}{\pi} \frac{1}{m+s} \frac{B_m^{(k_1, k_2 m \ldots, k_n)}}{m!}$$

$$= \frac{\Gamma(1+m)}{m!} \cos(-\pi m)\, B_m^{(k_1, k_2, \ldots, k_n)}$$

$$= (-1)^m B_m^{(k_1, k_2, \ldots, k_n)}.$$

Thus we have the conclusion:

The generalized multiple zeta function $\zeta_n(k_1, k_2, \ldots, k_n; s)$ is analytically continued to an entire function on the complex s-plane, and the special values at non-positive integers are given by the generalized Bernoulli numbers with order n, i.e.,

$$\zeta_n(k_1, k_2, \ldots, k_n; -m) = (-1)^m B_m^{(k_1, k_2, \ldots, k_n)}.$$

Remark : For $k_i > 1$ $(n+1 \le i \le 2n)$, $\mathrm{Re}(s) > 1$, recall the multiple zeta values :

$$\zeta(k_1, k_2, \ldots, k_n, k_{n+1}, \ldots, k_{2n}; s)$$

$$= \sum_{0 < m_1 < m_2 < \ldots < m_{2n-1}} \frac{1}{m_1^{k_1} m_2^{k_2} \ldots m_{2n-1}^{k_{2n-1}}} \sum_{m_{2n} = m_{2n-1}+1}^{\infty} \frac{1}{m_{2n}^{s}}$$

$$= \sum_{0 < m_1 < m_2 < \ldots < m_{2n-1}} \frac{1}{m_1^{k_1} m_2^{k_2} \ldots m_{2n-1}^{k_{2n-1}}} \sum_{m_{2n} = m_{2n-1}+1}^{\infty} \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-m_{2n} t} t^{s-1} \, dt$$

$$= \frac{1}{\Gamma(s)} \int_0^{\infty} \left(\sum_{0 < m_1 < m_2 < \ldots < m_{2n-1}} \frac{e^{-m_{2n-1} t}}{m_1^{k_1} m_2^{k_2} \ldots m_{2n-1}^{k_{2n-1}}} \right) \frac{t^{s-1}}{e^t - 1} \, dt$$

$$= \ldots$$

$$= \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{(e^t - 1)^n} Li_{k_1, k_2, \ldots, k_n}(e^{-t}) \, dt.$$

The above formula is closely related to the generalized multiple zeta functions.

RFERENCES

1. J. M. Borwein, D. M. Bradley and D. H. Broadhurst, *Electron. J. Combinatorics* **4** (1997), 2-21.
2. H. M. Srivastava and P.H. Todorov, *J. Nr. Theor.* **130** (1988), 503-13.