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SECOND ORDER DIFFERENTIAL INCLUSIONS WITH A MAXIMAL MONOTONE TERM AND A NONCONVEX, UNBOUNDED MULTIFUNCTION*

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In this paper we consider a second order multivalued periodic problem with a maximal monotone term and a nonconvex and unbounded orientor field. Using directionally continuous selectors we pass to a related convex problem which we solve via a fixed point argument. Then we show that the solutions of the convex problem also solve the original inclusion.

Key Words : Directionally Continuous Selector; Filippov Regularization; Unbounded Nonconvex Orientor Field; Multivalued Leray-Schauder Alternative Theorem; Compact Operator, Green's Identity; Density Point; Lusin's Theorem; Sobolev Embedding Theorem

List of Symbols

P_{fc} : P subscript fc,

λ : Lower case Greek letter lambda,

Γ : Upper case greek letter gamma,

η : Lower case greek letter eta,

Γ^{η}: Γ superscript η,

ξ : Lower case Greek letter xi,

ϕ : Lower case Greek letter phi,

$\hat{A}$: A hat.

1. INTRODUCTION

In this paper we study second order differential inclusions in $\boldsymbol{R}^N$, with a maximal monotone term, with an unbounded and nonconvex valued orientor field (multivalued term) and with periodic boundary value conditions. More precisely, let $T = [0, b]$. The problem under consideration is the

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following :

$$\left\{\begin{array}{l} x''(t) \in Ax(t) + F(t, x(t), x'(t)) \text{ a.e. on } T \\ x(0) = x(b), x'(0) = x'(b). \end{array}\right\} \qquad \ldots (1)$$

Here $A : \boldsymbol{R}^N \to 2^{\boldsymbol{R}^N} \setminus 0$ is a maximal monotone map. As it will become evident in the next sections, our analysis applied to the Dirichlet and Neumann problems.

Until now second order differential inclusions have been studied with $A \equiv 0$ and mostly with Dirichlet boundary conditions and with a multifunction $F(t, x, y)$ taking compact and convex values and satisfying a growth condition. There have been some works where the convexity of the values of F has been droped and\or the Dirichlet boundary conditions have been replaced by other boundary relations. These developments can be traced in the works of Erbe-Krawcewicz[2], Frigon-Granas[3], Kandilakis-Papageorgiou[8], Kravvaritis-Papageorgiou[9], Marano[10] and Pruszko[12]. However, all these works have $A \equiv 0$ and F satisfies certain growth conditions. So it seems that none of the previous works has examined second order multivalued boundary value problems in this generality.

In this paper we use the approach of Bressan[1] to pass from the nonconvex problem to a related convex one, whose solutions also solve the original inclusion. This approach was also used in the context of first order periodic differential inclusions in $\boldsymbol{R}^N$ with nonconvex right hand side by Hu-Papageorgiou[7] and Hu-Kandilakis-Papageorgiou[8].

2. Mathematical Preliminaries

In what follows, by $P_{f(c)}(R^N)$ we denote the set of all nonempty, closed (and convex) subsets of $\boldsymbol{R}^N$. A multifunction $F : T \times \boldsymbol{R}^N \times \boldsymbol{R}^N \to P_f(\boldsymbol{R}^N)$ is said to be measurable if for all $U \subseteq \boldsymbol{R}^N$ open, the set $F^{-(V)} = \left\{ (t, x, y) \in T \times \boldsymbol{R}^N \times \boldsymbol{R}^N : F(t, x, y) \cap U \neq 0 \right\} \in B(T) \times B(\boldsymbol{R})^N \times B(\boldsymbol{R}^N)$ with $B(T)$ (resp. $B(\boldsymbol{R}^N)$) being the Borel σ-field of T (resp. of $\boldsymbol{R}^N$)). This definition of measurability of $F(\cdot, \cdot, \cdot)$ is equivalent to saying that for all $v \in \boldsymbol{R}^N$, $(t, x, y) \to d(v, F(t, x, y)) = \inf [\| v - u \| : u \in F(t, x, y)]$ is a measurable $\overline{\boldsymbol{R}}_+$-valued function with $\overline{\boldsymbol{R}}_+ = \boldsymbol{R}_+ \cup \{+\infty\}$ (see Proposition 1.3, p. 142 of Hu-Papageorgiou[6]).

Let Y, Z be Hausdorff topological spaces. A multifunction $G : Y \to 2^Z - \{0\}$ is said to be upper semicontinuous (u.s.c.) (resp. lower semicontinuous (l.s.c.)) if for all $C \subseteq Z$ closed, the set $G^-(C) = \{y \in Y : G(y) \cap C \neq 0\}$ (resp. $G^+(C) = \{y \in Y : G(y) \subseteq C\}$ is closed in Y.

Next let Γ be a cone in $\boldsymbol{R}^N$ and Z a metric space. Following Bressan[1] we say that a single-valued map $f : \boldsymbol{R}^N \to Z$ is "Γ-continuous at $v \in \boldsymbol{R}^N$, if $f(v_n) \xrightarrow{n \to \infty} f(v)$ in Z when $v_n \xrightarrow{n \to \infty} v$ in $\boldsymbol{R}^N$ and $v_n - v \in \Gamma, n \geq 1$ (equivalently for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d_Z(f(v'), f(v)) < \varepsilon$ when $\| v' - v \| < \delta$ and $v' - v \in \Gamma$). We say that f is Γ-continuous, if it is Γ-continuous at every $v \in \boldsymbol{R}^N$. In our case the relevant cone will be $\Gamma^\eta = \left\{ (t, x, y) \in T \times \boldsymbol{R}^N \times \boldsymbol{R}^N : \| x \| + \| y \| \leq \eta t \right\}$.

Also let us recall the multivalued version of the Leray-Schauder alternative theorem :

Proposition 1 — If C is a convex subset of a Banach space $X, 0 \in X$ and $V : C \to 2^C \setminus \{0\}$ is a multifunction with nonempty, compact and convex values which maps bounded sets into relatively compact sets, then one and only one of the following two statements is true :

(a) $V(\cdot)$ has a fixed point in C (i.e. there exists $x \in C$ such that $x \in V(x)$);

(b) the set $\{x \in C$: there exists $0 < \lambda < 1$ with $x \in \lambda V(x)\}$ is unbounded.

Finally by $\|\cdot\|_{m,2}$ we denote the norm of the Sobolev space $W^{m,2}(T, \boldsymbol{R}^N)$; i.e.

$\|x\|_{m,2} = \left(\sum_{k=0}^{m} \|x^{(k)}\|_2^2 \right)^{\frac{1}{2}}$. Recall the by the Sobolev embedding theorem the space $W^{2,2}(T, \boldsymbol{R}^N)$ is embedded compactly in $C^1(T, \boldsymbol{R}^N)$. Thus we can find $\xi > 0$ such that $\|x\|_\infty + \|x'\|_\infty \leq \xi \|x\|_{2,2}$ for all $x \in W^{2,2}(T, \boldsymbol{R}^N)$.

3. AUXILIARY PROBLEM

In this section we implement the method of Bressan[1] and introduce a related "convex" problem which we solve.

First let us state our hypotheses on the data of the problem (1).

$H(A) : A : R^N \to 2^{R^N} \backslash \{0\}$ is a maximal monotone map with $0 \in A(0)$.

$H(F) : F : T \times R^N \times R^N \to P_f(R^N)$ is a measurable function such that :

(i) $(t, x, y) \to F(t, x, y)$ is measurable;

(ii) for all $t \in T$, $(x, y) \to F(t, x, y)$ is l.s.c.;

(iii) inf $[\|u\| : u \in F(t, x, y)] \leq \phi(t)$ a.e. on T with $\phi \in L^2(T)$; and

(iv) there exists $M > 0$ such that if $\|x_0\| > M$ and $(x_0, y_0)_{R^N} = 0$, we can find $\delta > 0$ such that for almost all $t \in T$ and all $\|x - x_0\| + \|y - y_0\| < \delta$ we have inf $[(u, x)_{R^N} + \|y\|^2 : u \in F(t, x, y)] > 0$.

Remark : Hypothesis $H(F)$ (iv) is an extension of the classical Nagumo-Hartman condition (see Hartman[4] p. 433) to the present multivalued and measurable context. We should emphasize that $H(F)$ above does not require $F(t, x, y)$ to be bounded and only a unilateral growth condition (from below) is imposed on F (see hypothesis $H(F)$ (iii)).

Let $\phi_1(t) = \phi(t) + 1$. Then $\phi_1 \in L^1(T)$. If $r > 0$, we set $B(0, r) = \{x \in R^N : \|x\| < r\}$.

Proposition 2 — If hypotnesis $H(f)$ hold and $F_1 : T \times \boldsymbol{R}^N \times \boldsymbol{R}^N \to P_f(\boldsymbol{R}^N)$ is defined by

$$F_1(t, x, y) = \overline{F(t, x, y) \cap B(0, \phi_1(t))}$$

then $(t, x, y) \to F_1(t, x, y)$ is measurable and for all $t \in T$ $(x, y) \to F(t, x, y)$ is l.s.c. .

PROOF : To prove the measurability of F_1 we need to show that for every $U \subseteq \boldsymbol{R}^N$ open, the set $G = \left\{ (t, x, y) \in T \times \boldsymbol{R}^N \times \boldsymbol{R}^N : F(t, x, y) \cap B(0, \phi_1(t)) \cap u \neq 0 \right\} \in B(T) \times B(\boldsymbol{R}^N) \times B(\boldsymbol{R}^N)$

(indeed recall that for any set $C \subseteq \boldsymbol{R}^N$, we have $C \cap U \neq 0$ if and only if $\overline{C} \cap u \neq 0$).

Note that $G = proj_{T\times \boldsymbol{R}^n \times \boldsymbol{R}^N}\left\{(t,x,y,u) \in T\times \boldsymbol{R}^N\times \boldsymbol{R}^N\times \boldsymbol{R}^N : u\ F(t,x,y)\cap U, \| u \| < \phi_1(t)\right\}$.

Let $K_1 = \left\{(t,x,y,u) \in t\times \boldsymbol{R}^N\times \boldsymbol{R}^N\times \boldsymbol{R}^N : u \in F(t,x,y)\cap U\right\}$. By virtue of hypothesis $H(F)$ (i) $(t,x,y) \to F(t,x,y)\cap U$ is measurable, thus graph measurable (see Hu-Papageorgiou[6], Corollary 1.9, p. 143). Thus $K_1 \in B(T)\times B(\boldsymbol{R}^N)\times B(\boldsymbol{R}^N)\times B(\boldsymbol{R}^N)$. Moreover by taking a Borel measurable version of the function $\phi_1(\cdot)$ wehave $K_2 = \left\{(t,x,y,u) \in t\times \boldsymbol{R}^N\times \boldsymbol{R}^N\times \boldsymbol{R}^N : \| u \| < \phi_1(t)\right\} \in B(T)\times B(\boldsymbol{R}^N)\times B(\boldsymbol{R}^N)\times B(\boldsymbol{R}^N)$. Then $K = K_1 \cap K_2 \in B(t)\times B(\boldsymbol{R}^N)\times B(\boldsymbol{R}^N)\times B(\boldsymbol{R}^N)$ and using Theorem 1.22, p. 147 of Hu-Papageorgiou[6] we have that $G = proj_{T\times R^N\times \boldsymbol{R}^N} \in B(T)\times B(\boldsymbol{R}^N)\times B(\boldsymbol{R}^N)$. This proves the measurability of the function F_1. Finally from Proposition 2.47, p 53 and Proposition 2.38, p. 50 of Hu-papageorgiou[6], it follows that $(x,y) \to F_1(t,x,y)$ is l.s.c. *Q.E.D.*

The above proposition allows us to apply the Scorza-Dragoni type result of Hu-Papageorgiou[7] (see Proposition 3.1) and produce $T_n \subseteq T, n \geq 1$, disjoint closed sets such that $T = \left(\bigcup_{n\geq 1} T_n\right) \bigcup T_0$ with $|T_0| = 0$ (here $|\cdot|$ denotes the Lebesgue measure on T), $F_1|_{T\times R^N\times R^N}$ is l.s.c. and $\phi_1 | T_n$ is continuous. Let $m_n = \max\left[\phi_1(t) : t \in T_n\right]$. Als since by hypothesis $H(A)$, A is a maximal monotone map which is defined everywhere on $\boldsymbol{R}^N$, we have that $A(\cdot)$ is locally bounded (see Hu-Papageorgiou[6], Theorem 1.21, p. 306). So if $M > 0$ is as is hypothesis $H(F)$ (iv), there exists $M_1 > 0$ such that $|A\,\overline{(B(0,M))}| = \sup\{\| y \| ; y \in \overline{A(B(0,M))}\} \leq M_1$. Let $\beta^2 = M^2 b + M_1 Mb + Mb^{1/2} \| \phi_1 \|_2 + 2M_1^2 b + 2\| \phi_1 \|_2^2$ and set $\eta_n = (1 + M_1 + m_n + \xi\beta), n \geq 1$, where $\xi > 0$ is such that $\| x \|_\infty + \| x' \|_\infty \leq \xi \| x \|_{2,2}$ for all $x \in W^{2,2}(T, \boldsymbol{R}^N)$ (see at the end of section 2). Using Theorem 2.1 of Bressan[1] (see also Hu-Papageorgiou[6], Theorem 5.2, p. 114), we can find a Γ^{η_n}-continuous selector $f_n(t,x,y)$ of the multifunction $F_1|_{T_n\times R^N\times R^N}$. The function $f_n : T_n\times R^N\times R^N \to R^N$ need to be continuous for the usual metric topology on $T_n\times R^N\times R^N$. For this reason we introduce its "Filippov regularization", namely the multifunction $G_n : T_n\times R^N\times R^N \to 2^{R^N}\setminus 0$ with compact and convex values defined by

$$G_n(t,x,y) = \bigcap_{\varepsilon>0} \overline{conv}\left[f_n(t',x',y') : |t-t'| < \varepsilon, \| x-x' \| < \varepsilon, \| y-y' \| < \varepsilon\right].$$

It is well known that $G_n(\cdot,\cdot,\cdot)$ is u.s.c. on $T_n\times R^N\times R^N$ (see Bressan[1] or Hu-Pagageorgiou[7]).

Now we define $G : T\times R^N\times R^N \to 2^{R^N}\setminus 0$ by

$$G(t,x,y) = \begin{cases} G_n(t,x,y) \text{ if } (t,x,y) \in T_n\times R^N\times R^N, n \geq 1. \\ (0) \text{ if } (t,x,y) \in T_0\times R^N\times R^N. \end{cases}$$

Evidently $t \to G(t, x, y)$ is measurable, $(x, y) \to G(t, x, y)$ is u.s.c. and the values of G are compact and convex. Moreover, it is clear from this construction that $|G(t, x, y)| = \sup [\|y\| : y \in G(t, x, y)] \leq \phi_1(t)$ for almost all $t \in T$ and all $x, y \in R^N$.

Now we consider the following auxiliary convex problem :

$$\left\{ \begin{array}{l} x''(t) \in Ax(t) + G(t, x(t), x'(t)) \text{ a.e. on} T \\ x(0) = x(b), x'(0) = x'(b). \end{array} \right\} \qquad \dots (2)$$

Next we solve problem (2). We start with some straightforward auxiliary results.

Let

$$D = \left\{ x \in W^{2,2}(T, R^N) : x(0) = x(b), x'(0) = x'(b) \right\}$$

and let

$$\hat{L} : D \subseteq L^2(T, R^N) \to L^2(T, R^N) \text{ be defined by } \hat{L}(x) = -x''. \text{ Set } L = \hat{L} + I.$$

Lemma 3 — $L : D \subseteq L^2(T, \boldsymbol{R}^N) \to L^2(T, R^N)$ is one-to-one and onto (i.e. bijective).

PROOF : Suppose $L(x_1) = L(x_2)$. Then $-x_1''(t) + x_1(t) = -x_2''(t) + x_2(t)$ a.e. on T and so $x_2''(t) - x_1''(t) - x_1(t)$ a.e. on T. It is, then, well known that $x_2(t) - x_1(t) = (c_1 e^t + c_2 e^{-t}) v$ where c_1, c_2 are arbitrary constants and $v = (1, 2, \dots, 1) \in R^N$. As $x_2 - x_1 \in D$ we have

$$\left\{ \begin{array}{l} c_1 + c_2 = c_1 e^b + c_2 e^{-b} \\ c_1 - c_2 = c_1 e^b - c_2 e^{-b} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} c_1 = c_1 e^b \\ c_2 = c_2 e^{-b} \end{array} \right\}$$

i.e., as $b \neq 0$ we have $x_1 = x_2$ in $L^2(T, R^N)$. So L is one-to-one.

Also if $h \in L^2(T, R^N)$, then from Seda[13] we know that

$$\left\{ \begin{array}{l} -x''(t) + x(t) = h(t) \text{ a.e. on } T \\ x(0) = x(b), x'(0) = x'(b) \end{array} \right\}$$

has a unique solution $x \in W^{2,2}(T, R^N)$ given by $x(t) = \int_0^b G(t, s)\, h(s)\, ds$, with $G(t, s)$ being the Green's function of the operator (L, D). *Q.E.D.*

By virtue of Lemma 3, the map $L^{-1} : L^2(T, R^N) \to D \subseteq W^{1,2}(T, R^N)$ is well defined. The next lemma gives us a useful property of the linear operator L^{-1}.

Lemma 4 — $L^{-1} : L^2(T, R^N) \to W^{1,2}(T, R^N)$ is compact.

PROOF : It suffices to show that if $y_n \xrightarrow{w} y$ in $L^2(T, R^N)$ then $L(y_n) \to L(y)$ in $W^{1,2}(T, R^N)$. Let $x_n = L(y_n), n \geq 1$ Then $x_n \in D \subseteq W^{1,2}(T, R^N)$ and $-x_n'' + x_n = y_n, n \geq 1$. Taking the inner product in $L^2(T, R^N)$ with x_n, we obtain

$$\int_0^b (-x_n''(t), x_n(t))_{R^N}\, dt + \| x_n \|_2^2 \le \| y_n \|_2 \| x_n \|_2.$$

Using Green's identity we have $\int_0^b (-x_n''(t), x_n(t))_{R^N}\, dt = \| x' \|_2^2$. So we have

$$\| x_n' \|_2^2 + \| x_n \|_2^2 = \| x_n \|_{1,2}^2 \le \| y_n \|_2 \| x_n \|_2 \qquad \ldots (3)$$

where by $\| \cdot \|_{1,2}$ we denote the norm in the Sobolev space $W^{1,2}(T, R^N)$. From (3) we infer at once that $\{x_n\}_{n \ge 1} \subseteq W^{1,2}(T, R^N)$ is bounded. Since $x_n'' = x_n - y_n, n \ge 1$, we see that $\{x_n\}_{n \ge 1} \subseteq D \subseteq W^{2,2}(T, R^N)$ is bounded. Thus we may assume that $x_n \xrightarrow{w} x$ in $W^{2,2}(T, R^N)$. In the limit we have $-x'' + x = y$. Also because $W^{2,2}(T, R^N)$ is embedded compactly in $C^1(T, R^N)$, we have $x_n \xrightarrow{n \to \infty} x$ in $C^1(T, R^N)$. Thus in the limit we have $x(0) = x(b), x'(0) = x'(b)$. Hence $x = L(y)$ and $x_n = L(y_n) \xrightarrow{n \to \infty} x = L(y)$ in $W^{1,2}(T, R^N)$ (by the compact embedding of $W^{2,2}(T, R^N)$ into $W^{1,2}(T, R^N)$). This proves the compactness of L^{-1}. *Q.E.D.*

Because of hypothesis $H(A)$, the map $A : R^N \to 2^{R^N} \setminus 0$ is u.s.c. and has closed and convex values (see Hu-Papageorgiou[6], Theorem 1.28, p. 308 and Proposition 1.14, p. 304). Let $\hat{A} : \hat{D} \subseteq L^2(T, R^N) \to 2^{L^2(T, R^N)}$ be the realization of A on the Hilbert space $L^2(T, R^N)$; i.e. $\hat{A}(x) = \{ h \in L^2(T, R^N) : h(t) \in A(x,(t)) \text{ a.e. on } T \}$ for all $x \in \hat{D} = \{x \in L^2(T, R^N)$: there exists $h \in L^2(T, R^N)$ such that $h(t) \in A(x,(t))$ a.e.on $T\}$. Note that $W^{1,2}(T, R^N) \subseteq \hat{D}$. Indeed if $x \in W^{1,2}(T, R^N)$ then $x \in C(T, R^N)$ and so since $A(\cdot)$ is locally bounded we have $|A(x(t))| = \sup [\| y \| : y \in A(x, t))] \le M_2$ for some M_2. So by the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou[6], Theorem 2.14, p. 158), we can find $h \in L^2(T, R^N)$ such that $h(t) \in A(x(t))$ a.e. on T. Also $\hat{A}(\cdot)$ is the maximal monotone (se Hu-Papageorgiou[6], Example 2.33, p. 328). Let $N : W^{1,2}(T, R^N) \to 2^{L^2(T, R^N)}$ be defined by $N(x) = \{g \in L^2(T, R^N) : g(t) \in G(t, x(t), x'(t))\}$. From Kandilakis-Papageorgiou[8] we know that $N(\cdot)$ has nonempty, closed and convex values and is u.s.c. from $W^{1,2}(T, R^N)$ into $L^2(T, R^N)$ furnished with the weak topology. Let $V = -\hat{A} - N + I$. Then problem (2) is equivalent to the following abstract multivalued fixed point problem

$$x \in L^{-1} V(x). \quad \dots (4)$$

We shall solve (4) using the multivalued Leray-Schauder alternative theorem (see Proposition 1).

Proposition 5 — If hypothesis $H(A)$ and $H(F)$ hold, then problem (2) has a solution $x \in W^{2,2}(T, R^N)$.

PROOF : As we already indicated we shall solve the equivalent fixed point problem (4) using Proposition 1. So first we show that $V: W^{1,2}(T, R^N) \to P_{fc}(L^2(T, R^N))$ is u.s.c. from $W^{1,2}(T, R^N)$ into $L^2(T, R^N)$ furnished with the weak topology. To this end let $C \subseteq L^2(T, R^N)$ be nonempty and weakly closed. We have to show that $V^-(C) = \left\{ x \in W^{1,2}(T, R^N) : V(x) \cap C \neq 0 \right\}$ is closed. So let $x_n \in V^-(C), n \geq 1$, and assume that $x_n \xrightarrow{n \to \infty} x$ in $W^{1,2}(T, R^N)$. Let $u_n \in V(x_n) \cap C, n \geq 1$. We have $-u_n = h_n + g_n - x_n$ with $h_n \in \hat{A}(x_n), g_n \in N(x_n), n \geq 1$. Since $x_n \xrightarrow{n \to \infty} x$ in $W^{1,2}(T, R^N)$, we have $x_n \xrightarrow{n \to \infty} x$ in $C(T, R^N)$ (from the continuous - in fact compact - embedding of $W^{1,2}(T, R^N)$ into into $C(T, R^N)$). So $\sup_{n \geq 1} \| x_n \|_\infty \leq M_3$ for some $M_3 \geq 0$. Hence $\| h_n(t) \| \leq | A\overline{(B(0, M_3))} | \leq M_4$ a.e. on T for some $M_4 > 0$. Also $\| g_n(t) \| \leq \phi_1(t)$ a.e. on T. Thus we may assume that $h_n \xrightarrow{w} h$ and $g_n \xrightarrow{w} g$ in $L^2(T, R^N)$. Moreover, since $x_n \xrightarrow{N \to \infty} x$ in $W^{1,2}(T, R^N)$, we may also assume that $x_n(t) \xrightarrow{n \to \infty} x(t)$ for all $t \in T, x_n'(t) \xrightarrow{n \to \infty} x'(t)$ a.e. on T and $u_n \xrightarrow{w} u$ in $L^2(T, R^N)$. Invoking Proposition 3.4, p. 692 of Hu-Papageorgiou[6], we have that

$$h(t) \in \overline{conv}\, \overline{\lim}\, A(x_n(t)) \subseteq A(x(t)) \text{ a.e. on } T$$

and

$$g(t) \in \overline{conv}\, \overline{\lim}\, G(t, x_n(t), x_n'(t)) \subseteq G(t, x(t), x'(t)) \text{ a.e. on } T,$$

the last inclusions in both relations being a consequence of the upper semicontinuity of the $P_{fc}(R^N)$ -valued multifunctions $A(\cdot)$ and $G(t, \cdot, \cdot)$. Thus in the limit as $n \to \infty$, we have that $-u = h + g - x$ with $h \in \hat{A}, g \in N(x)$. Also $u \in C$ since C is weakly closed in $L^2(T, R^N)$. Therefore $u \in V(x) \cap C$, hence $V^-(C)$ is closed in $W^{1,2}(T, R^N)$, which proves the upper smicontinuity of V from $W^{1,2}(T, R^N)$ into $L^2(T, R^N)$ with the weak topology. Moreover, it is clear that V maps bounded sets in $W^{1,2}(T, R^N)$ into bounded sets in $L^2(T, R^N)$. Now because of Lemma 4 $L^{-1} V: W^{1,2}(T, R^N) \to P_{fc}(W^{1,2}(T, R^N)$ is u.s.c. (see Hu-Papageorgiou[6]], Proposition 2.56, p. 57) and maps bounded sets into relatively compact sets. So in order to use Proposition 1 and solve (4) (hence (2) too) we need to show that $\Gamma = \left\{ x \in D \subseteq W^{1,2}(T, R^N) : x \in \lambda L^{-1} V(x), 0 < \lambda < 1 \right\}$ is bounded in $W^{1,2}(T, R^N)$. To this end let $x \in \Gamma$. Then for some $0 < \lambda < 1$ we have

$$L\left(\frac{1}{\lambda}x\right) = -h - g + x \text{ for some } h \in \hat{A}, g \in N(x)$$

$$\Rightarrow -x''(t) = -\lambda h(t) - \lambda g(t) + (\lambda - 1)x(t) \text{ a.e. on } T.$$

Take the inner product with $x(t)$ and the integrate over T. Using Green's identity, we have

$$\|x'\|_2^2 \leq -\lambda \int_0^b (h(t), x(t))_{R^N} dt - \lambda \int_0^b (g(t), x(t))_{R^N} dt \text{ (since } 0 < \lambda < 1)$$

$$\leq \|h\|_2 \|x\|_2 + \|\phi_1\|_2 \|x\|_2. \qquad \ldots (5)$$

Claim — For every $x \in \Gamma$, we have $\|x\|_\infty \leq M$ with $M > 0$ as in hypothesis $H(F)$ (iv).

PROOF OF CLAIM — We start with the straightforward observation, that $G(t, x, y)$ too satisfies hypothesis $H(F)$ (iv). Then let $r(t) = \|x(t)\|^2$ and let $t_0 \in T$ be a point where $r(\cdot)$ attains its supremum. Suppose that $r(t_0) > M^2$. First assume that $0 < t_0 < b$. Then $r'(t_0) = 2(x'(t_0), x(t_0))_R^N = 0$. Since as we already mentioned G satisfies hypothesis $H(F)$ (iv), we can find $c, \delta > 0$ such that for almost all $t \in T$

$$\inf\left[(u, x)_{R^N} + \|y\|^2 : u \in G(t, x, y), \|x(t_0) - x\| + \|x'(t_0) - y\| < \delta\right] \geq c.$$

Since $x \in \Gamma \subseteq W^{2,2}(T, R^N)$ embeds continuously into $C^1(T, R^N)$, we can find $\delta_1 > 0$ such that if $t_0 \leq t \leq t_0 + \delta_1$, we have

$$\|x(t_0) - x(t)\| + \|x'(t_0) - x'(t)\| < \delta.$$

Also $g(t) \in G(t, x(t), x'(t))$ a.e. on T. So we have for almost all $t \in (t_0, t_0 + \delta_1]$

$$\lambda(g(t), x(t))_{R^N} + \lambda \|x'(t)\|^2 \geq \lambda c. \qquad \ldots (6)$$

But $\lambda g(t) \in x''(t) - \lambda Ax(t) + (\lambda - 1)x(t)$ a.e. on T. Taking the inner product with $x(t)$ and using (6) and the facts that $0 \in A(0)$ and that $0 < \lambda < 1$, we have for almost all $t \in (t_0, t_0 + \delta_1]$

$$\lambda_c \leq (x''(t), x(t))_{R^N} + \lambda \|x'(t)\|^2$$

$$\Rightarrow \lambda c(t - t_0) \leq (x'(t), x(t))_{R^N} - \int_{t_0}^t \|x'(s)\|^2 ds + \lambda \int_{t_0}^t \|x'(s)\|^2 ds$$

(integrating, using Green's identity and recalling that $(x'(t_0), x(t_0))_{R^N} = 0$). Since $0 < \lambda < 1$, we have

$$\lambda c(t - t_0) \leq (x'(t), x(t))_{R^N} \text{ for almost all } t \in (t_0, t_0 + \delta_1]$$

$$\Rightarrow r'(t) > 0 \text{ for almost all } t \in (t_0, t_0 + \delta_1],$$

which contradicts to the choice of t_0. Therefore $\| x(t_0 \| \leq M$ when $0 < t_0 < b$. Next assume $t_0 = 0$. Then $r'(0) \leq 0$. By periodicity $t = b$ is also a maximizer for $r(\cdot)$ and so $r'(b) \geq 0$. But $r'(0) = r'(b)$ (due to the periodic boundary conditions). Hence $r'(0) = 0$ and proceed as before. Similarly if $t_0 = b$. So in all cases we have that $\| x(t) \| \leq M$ for all $t \in T$ and all $x \in \Gamma$. This proves the claim.

We return to inequality (5) and obtain first that $\| h(t) \| \leq | \overline{(B(0, M))} | \leq M_1$ and then that

$$\| x' \|_2^2 \leq Mb^{\frac{1}{2}} + \| \phi_1 \|_2) = M_5 \text{ for all } x \in \Gamma.$$

Therefore this bound combined with the claim, tell us that Γ is bounded in $W^{1,2}(T, R^N)$. Invoking Proposition 1, we obtain $x \in D$ such that $x \in L^{-1} V(x)$. Then clearly this $x \in W^{2,2}(T, R^N)$ solves problem 92). *Q.E.D.*

4. Main Theorem

In this section we show that problem (1) has a solution. More precisely we show that every solution of (2) also solves (1) and since (2) has a nonempty solution set (see Proposition 5), we have the desired existence theorem for problem (1). First a simple auxiliary result :

Lemma 6 — If $x(\cdot) \in D$ is a solution of (2) and $S = \{t \in T : x''(t) \in Ax(t) + G(t, x(t), x'(t))$ and there exists a strictly decreasing sequence $\{t_m\}_{m \geq 1} \subseteq T$ such that $t_m \downarrow t$, $x''(t_m) \to x''(t)$ **and $x''(t_m) \in Ax(t_m) + G(t_m, x(t_m), x'(t_m))$ for all $m \geq 1\}$, then $|S| = |T|$.**

Proof : Lusin's theorem tells us that given $\varepsilon > 0$, we can find $T_\varepsilon \subseteq T$ closed such that $|T| - \varepsilon \leq |T|$ and $x''|_{T_\varepsilon}$ is continuous. Let $S_\varepsilon = \{t \in T : x''(t) \in Ax(t) + G(t, x(t), x'))\} \cap T_\varepsilon$. Evidently, $|S_\varepsilon| \geq |T| - \varepsilon$. Let $S'_\varepsilon = \{t \in S_\varepsilon : t \text{ is a point of density}\}$. From Lebesgue's theorem (see Oxtoby[11], Theorem 3.20, p. 17) we know that $|S'_\varepsilon| = |S_\varepsilon|$. But then by definition for every $t \in S'_\varepsilon$ we can find $\{t_m\}_{m \geq 1} \subseteq S_\varepsilon$ strictly decreasing, such that $t_m \downarrow t$. Hence $|S| \geq |S'_\varepsilon| = |S_\varepsilon| \geq |T| - \varepsilon$. Let $\varepsilon \downarrow 0$ to conclude that $|S| = |T|$. *Q.E.D.*

Now we are ready for the existence theorem concerning problem (1).

Theorem 7 — *If hypotheses H (A) and H(F) hold, then problem (1) has a solution* $x \in W^{2,2}(T, R^N)$.

PROOF : Let $x \in D \subseteq W^{2,2}(T, R^N)$ be a solution of problem (2). Its existence is guaranteed by Proposition 5. In what follows we shall show that $x''(t) \in Ax(t) + f_n(t, x(t), x'(t))$ a.e. on T and so $x(\cdot)$ is the desired solution of problem (1).

Let $S_n = \{t \in T_n : x''(t) \in Ax(t) + G(t, x(t), x'(t))$ and there exists a strictly decreasing sequence $\{t_m\}_{m \geq 1} \subseteq T_n$ such that $t_m \downarrow t$, $x''(t_m) \to x''(t)$ as $m \to \infty$ and $x''(t_m) \in Ax(t_m) + G(t_m, x(t_m), x'(t_m))$ for all $m \geq 1\}$. Lemma 6 tells us that $|S_n| = |T_n|$ for all $n \geq 1$. Let $t \in S_n$ and fix $\varepsilon > 0$. Since f_n is Γ^{η_n}-continuous, we can find $\delta > 0$ such that for all $(t', z, y) \in T_n \times R^N \times R^N$ with $t < t' < t + \delta$ and $\|z - x(t)\| + \|y - x'(t)\| < \eta_x(t' - t)$, we have $\|f_n(t', z, y) - f_n(t, x(t), x'(t))\| \leq \varepsilon$. Since $t \in S_n$, from the definition of the latter, we can find $\{t_m\}_{m \geq 1} \subseteq T_n$ a strictly decreasing sequence such that $t_m \downarrow t$ and $x''(t_m) \to x''(t)$, $x''(t_m) \in Ax(t_m) + G(t_m, x(t_m), x'(t_m))$. Thus we can find $m_0 \geq 1$ such that for $m \geq m_0$ we have

$$t < t_m < t + \delta, \left\| \frac{x'(t_m) - x'(t)}{t_m - t} - x''(t) \right\| \leq 1, \|x''(t_m) - x''(t)\| < \varepsilon$$

and

$$x''(t_m) \in Ax(t_m) + G(t_m, x(t_m), x'(t_m)) .$$

We have for $m \geq m_0$:

$$\|x'(t_m) - x'(t)\| = \left\| \frac{x'(t_m) - x'(t)}{(t_m - t)} \right\| (t_m - t) \leq \| \frac{x'(t_m) - x'(t)}{t_m - t}$$

$$- x''(t)\| (t_m -) + \|x''(t)\| (t_m - t) \leq (1 + \|x''(t)\|)(t_m - t).$$

But $x''(t) \in Ax(t) + G(t, x(t), x'(t)) \Rightarrow \|x''(t)\| \leq |A\overline{(A(B(0, M))}| + \phi_1(t) \leq M_1 + m_n$ (since for all solutions $x \in D$ of (2) we have $\|x\|_\infty \leq M$, see the proof of Proposition 5, and by the definition $m_n = \sup\,[\phi_1(t) : t \in T_n]$). Thus

$$\|x'(t_m) - x'(t)\| \leq (1 + M_1 + m_n)(t_m - t).$$

Also since for all $t \in T$ we have $\|x(t)\| \leq M$, we obtain $\|x\|_2^2 = \int_0^b \|x(t)\|^2 dt \leq M^2 b$. From (5) we know that $\|x'\|_2^2 \leq \int_0^b \|h(t)\| \|x(t)\| dt + \int_0^b \|g(t)\| \|x(t)\| dt \leq M_1 Mb + Mb^{\frac{1}{2}} \|\phi_1\|_2^2 = \beta^2$

$$\Rightarrow \| x' \|_\infty \leq \xi \beta.$$

From the mean value theorem, we have that

$$\| x(t_m) - x(t) \| \leq (t_m - t) \ \sup \ [\| x'(x) \| : t < c < t_m] \leq (t_m - t) \, \xi \beta.$$

Hence, finally we have

$$\| x(t_m) - x(t) \| + \| x'(t_m) - x'(t) \| \leq (1 + M_1 + m_n + \xi \beta)(t_m - t) = \eta_n (t_m - t)$$

$$\Rightarrow \| f_n(t_m, x(t_m), x'(t_m)) - f_n(t, x(t), x'(t)) \| < \varepsilon$$

$$\Rightarrow G(t_m, x(t_m), x'(t_m)) \in f_n(t, x(t), x'(t)) + 2\varepsilon B(0, 1)$$

Also because $A(\cdot)$ is u.s.c., we can find $m_1 \geq 1$ such that for $m \geq m_1$ we have

$$Ax(t_m) \subseteq Ax(t) + \varepsilon B(0, 1).$$

Let $m_2 = max \ \{m_0, m_1\}$ and $m \geq m_2$. We have

$$x''(t_m) \in Ax(t_m) + G(t_m, x(t_m), x'(t_m)) \subseteq Ax(t) + f_n(t, x(t), x'(t)) + 3 \varepsilon B(0, 1).$$

Also $\| x''(t_m) - x''(t) \| < \varepsilon$, thus $x''(t) \in x''(t_m) + \varepsilon B(0, 1)$. Hence,

$$x''(t) \in Ax(t) + f_n(t, x(t), x'(t)) + 4\varepsilon B(0, 1).$$

Let $\varepsilon \downarrow 0$ to conclude that

$$x''(t) \in Ax(t) + f_n(t, x(t), x'(t)) \subseteq Ax(t) + F(t, x(t), x'(t)) \text{ a.e. on } T_n \text{ for all } n \geq 1$$

$\Rightarrow x(\cdot)$ is the desired solution of (1). *Q.E.D.*

Remark : It is clear that crucial in our arguments was the hypothesis that $D(A) = R^N$. It will be interesting to know if we can have Theorem 7 without this hypothesis. Another interesting open problem is to see whether a result like Theorem 7 holds for quasilinear problems of the form $(\| x'(t) \|^{p-2} x'(t))' \in Ax(t) + F(t, x(t), x'(t))$ a.e. on $T, x(0) = x(b), x'(0) = x'(b)$ with $2 \leq p < \infty$. Finally a careful reading of our arguments in this paper reveals that the same proofs with minor eacy modifications apply also to the Neumann problem (and of course to the easier Dirichlet problem).

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