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A VECTOR EXTENSION TO BEHERA AND PANDA'S GENERALIZATION OF MINTY'S LEMMA

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In this paper, we consider a generalized result of Behera and Panda for Minty's lemma, and obtain an existence theorem of solution to vector variational-type inequality.

Key Words : KKM-Fan Theorem; Generalized Minty's Lemma; Vector Variational-Type inequality cone

1. INTRODUCTION

Giannessi[6] was the firstto introduce the vector variational inequality in a finite-dimensional Euclidean space in 1966, which is the vector-valued version of the variational inequality of Hartman and Stampacchia[8]. Later on, many authors (see, Chen *et al.*[3 & 4], Konnov *et al.*[10], Lee *et al.*[12, 13, 14], Lee *et al.*[16, 17], Park *et al.*[20], Siddiqi *et al.*[21] and Yu[22]) have investigated vector variational inequalities in abstract spaces.

Minty[18] proved the linearization lemma, which played a useful role in variational inequalities. In fact, the classical Minty's inequality and Minty's lemma have been shown to be an important tool in the regularity results of the solution for a generalized non-homogeneous boundary value problem[1] and, when the operator is a gradient, also a minimum principle for convex optimization problems[9]. On the other hand, vector variational inequality is closed related to vector optimization problem. Giannessi[7] established the equivalence between a differentiable convex vector optimization problem and a vector variational inequality can be an efficient tool for studying vector optimization problems. Moreover, using a vector variational-like inequality, Lee *et al.*[15] proved existence theorems for solutions of nondifferentiable invex optimization problems.

Recently, Behera and Panda[2] obtained a nonlinear generalization of Minty's lemma. Furthermore they applied the result to obtain a solution of a certain variational-like inequality.

In 1999, Lee *et al.*[12] obtained a vector version of Minty's lemma using Nadler's result[19], and with their result they considered two kinds of vector variational-like inequalities for multifunctions under certain new pseudomonotonicity condition and certain new hemicontinuity condition, respectively, different from conditions in [14 & 21].

In this paper, we first consider a generalized result of Behera and Panda for Minty's lemma by extending to the vector case under traditional pseudomonotonicity and traditional hemicontinuity

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conditions. And then with the result, we consider the existence of solutions to some vector variational-type inequalities.

2. PRELIMINARIES

Let X, Y be topological vector spaces, K be a nonempty convex subset of X, and $T: K \to L(X, Y)$, $\theta: K \times K \to X$, $\eta: K \times K \to Y$ be operators, where $L(X, Y)$ is the space of all linear continuous operators from X into Y. Let $\{C(x): x \in K\}$ be a family of closed convex cones in Y and *int* denotes the interior. We define a relation $\leq_{C(x)}$ in Y by the convex cone $C(x)$[11]; for $y_1, y_2 \in Y, y_1 \leq_{C(x)} y_2$ if and only if $y_2 - y_1 \in C(x)$.

An operator $T: K \to L(X, Y)$ is said to be finite dimensional continuous[2] if for every finite dimensional subspace M of X, the operator $T: K \cap M \to Y$ is weakly continuous.

Let $T: K \to L(X, Y)$ be an operator. T is said to be hemicontinuous[22] if for any $x, y \in K$, $\alpha \in [0, 1]$, the mapping

$$\alpha \mapsto \langle T(\alpha y + (1-\alpha) x), y - x \rangle$$

is continuous.

A mapping $f: K \to Y$ is convex[11] if for every $x_1, x_2 \in K$ and $\lambda \in (0, 1)$,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq_{C(x)} \lambda f(x_1) + (1-\lambda) f(x_2),$$

i.e.,

$$\lambda f(x_1) + (1-\lambda) f(x_2) - f(x_2) - f(\lambda x_1 + (1-\lambda)x_2) \in C(x).$$

Let K be a subset of a topological vector space X. Then a mapping $T: K \to 2^X$ is called Knaster-Kuratowski-Mazurkiewicz (in short, KKM) mapping[13] if for each nonempty finite subset N of K, $co\, N \subset F(N)$, where co denotes the convex hull and $F(N) = \bigcup \{F(x): x \in N\}$.

The following theorem will be used to prove our main theorem in the next section.

Theorem 2.1 (Fan[5]) — *Let K be an arbitrary nonempty subset of a Hausdorff topological vector space. Let a set-valued mapping $T: K \to 2^X$ be a KKM-mapping such that $F(x)$ is closed for all $x \in K$ and compact for at least one $x \in K$. Then*

$$\bigcap_{x \in K} F(x) \neq 0.$$

3. MAIN RESULTS

Now we consider a vector generalization of Behera and Panda's result for Minty's lemma.

Theorem 3.1 — *Let X, Y be topological vector spaces. Let K be a nonempty convex subset of X and $\{C(x): x \in K\}$ be a family of closed convex cones in Y. Let a set-valued mapping $W: K \to 2^Y$, defined by $W(x) = Y \backslash \{-\, int\, C(x)\}$, have a closed graph. Assume that $T: K \to L(X, Y)$, $\theta: K \times K \to X$ and $\eta: K \times K \to Y$ are operators such that —*

(1) $\langle T(y), \theta(y, y) \rangle + \eta(y, y) \in C(x)$, for all $x, y \in K$.

(2) The operator

$$y \mapsto \langle T(y), \theta(x, y) \rangle + \eta(y, x)$$

of K into Y is finite dimensional continuous (or at least hemicontinuous) for all $x \in K$.

(3) The operator

$$x \mapsto \langle T(y), \theta(x, y) \rangle + \eta(y, x)$$

of K into Y is convex for all $y \in K$.

(4) $\langle T(x), \theta(y, x) \rangle + \eta(x, y) \notin -\, int\, C(x)$,

for all $y \in K$ implies

$$\langle T(y), \theta(x, y) \rangle + \eta(y, x) \notin int\, C(x), \text{ for all } y \in K.$$

Then the following are equivalent :

(a) there exists an $x_0 \in K$ such that

$$\langle T(x_0), \theta(y, x_0) \rangle + \eta(x_0, y) \notin -\, int\, C(x_0), \text{ for all } y \in K; \text{ and}$$

(b) there exists an $x_0 \in K$ such that

$$\langle T(y), \theta(x_0, y) \rangle + \eta(y, x_0) \notin -\, int\, C(x_0), \text{ for all } y \in K.$$

PROOF : Suppose there exists an $x_0 \in K$ such that $\langle T(x_0), \theta(y, x_0) \rangle + \eta(x_0, y) \notin -\, int\, C(x_0)$ for all $y \in K$, then by the condition (4), we have $\langle T(y), \theta(x_0, y) \rangle + \eta(y, x_0) \notin int\, C(x_0)$. Conversely, suppose there exists an $x_0 \in K$ such that $\langle T(y), \theta(x_0, y) \rangle + \eta(y, x_0) \notin int\, C(x_0)$ for all $y \in K$. For any arbitrary $x \in K$, let $y_t = tx + (1-t)x_0$, $t \in (0, 1)$ then $y_t \in K$, since K is convex. Thus,

$$\langle T(y_t), \theta(x_0, y_t) \rangle + \eta(y_t, x_0) \notin int\, C(x_0). \qquad \ldots (3.1)$$

On the other hand, by convexity of the operator,

$$\{\langle T(y_t), \theta(y_t, y_t) \rangle + \eta(y_t, y_t)\}$$

$$\leq_{C(x_0)} t\{\langle T(y_t), \theta(x, y_t) \rangle + \eta(y_t, x)\} + (1-t)\{\langle T(y_t), \theta(x_0, y_t) \rangle + \eta(y_t, x_0)\}.$$

So, we get

$$t\{\langle T(y_t), \theta(x, y_t) \rangle + \eta(y_t, x)\} + (1-t)\{\langle T(y_t), \theta(x_0, y_t) \rangle + \eta(y_t, x_0)\}$$

$$- \{\langle T(y_t), \theta)y_t, y_t) \rangle + \eta(y_t, y_t)\} \in C(x_0).$$

By condition (1), it follows that

$$t\{\langle T(y_t), \theta(x, y_t) \rangle + \eta(y_t, x)\} + (1-t)\{\langle T(y_t), \theta(x_0, y_t) \rangle + \eta(y_t, x_0)\} \in C(x_0).$$

Now, we show that $\langle T(y_t), \theta(x, y_t) \rangle + \eta(y_t, x) \notin -Int\, C(x_0)$. In fact, suppose that $\langle T(y_t), \theta(x, y_t) \rangle + \eta(y_t, x) \in -Int\, C(x_0)$, then $\langle T(y_t), \theta(x_0, y_t) \rangle + \eta(y_t, x_0) \in Int\, C(x_0)$, which contradicts (3.1). Thus, we have

$$\langle T(y_t), \theta(x, y_t) \rangle + \eta(y_t, x) \notin -int\, C(x_0). \qquad \text{... (3.2)}$$

Since the operator

$$y \mapsto \langle T(y), \theta(x, y) \rangle + \eta(y, x)$$

of K into Y is finite dimensional continuous (or at least hemicontinuous), taking the limit as t approaches to 0^+ in (3.2), we get

$$\langle T(x_0), \theta(x, x_0) \rangle + \eta(x_0, x) \notin -int\, C(x_0).$$

Since x is arbitrary, hence

$$\langle T(x_0), \theta(y, x_0) \rangle + \eta(x_0, y) \notin -int\, C(x_0).$$

Corollary 3.2 — When X is a reflexive real Banach space, $L(X, Y) = X^*$, $Y = \mathbb{R}$, $C(x) = \mathbb{R}^+$, if η is a zero functional, then we obtain Theorem 4.1 in [2] as a corollary.

Next we apply Theorem 3.1 to prove the existence of a solution to the following vector variational-type inequality.

Theorem 3.3 — *Let X be a reflexive real Banach space and Y a topological vector space, K a nonempty closed convex and bounded subset of X and $\{C(x) : x \in K\}$ a family of closed convex cones in Y. Let a set-valued mapping $W : K \to 2^Y$, defined by $W(x) = Y \;\{- intC(x)\}$, have a closed graph. Assume that $T : K \to L(X, Y)$, $\theta : K \times K \to X$ and $\eta : K \times K \to Y$ are operators such that —*

(1) the operator

$$x \mapsto \langle T(y), \theta(x, y) \rangle + \eta(y, x)$$

of K into Y is convex for all $y \in K$;

(2) the operator

$$x \mapsto \langle T(y), \theta(x, y) \rangle + \eta(y, x)$$

of K into Y is continuous for all $y \in K$; and

(3) $\langle T(x), \theta(y, x) \rangle + \eta(x, y) \notin -int\, X(x)$, for all $y \in K$ implies $\langle T(y), \theta(x, y) \rangle + \eta(y, x) \notin int\, C(x)$, for all $y \in K$.

Then there exists an $x_0 \in K$ such that

$$\langle T(y), \theta(x_0, y) \rangle + \eta(y, x_0) \notin int\, C(x_0),$$

for all $y \in K$.

Moreover if the following conditions hold

(4) $\langle T(y), \theta(y, y) \rangle + \eta(y, y) \in C(x)$, for all $x, y \in K$ and

(5) the operator

$$y \mapsto \langle T(y), \theta(x, y) \rangle + \eta(y, x)$$

of K into Y is finite dimensional continuous (or at least hemicontinuous) for all $x \in K$, then there exists an $x_0 \in K$ such that

$$\langle T(x_0), \theta(y, x_0) \rangle + \eta(x_0, y) \notin -int\, C(x_0), \text{ for all } y \in K.$$

PROOF : Define a set-valued mapping $F : K \to 2^K$ by

$$F(y) := \{x \in K : \langle T(x), \theta(y, x) \rangle + \eta(x, y) \notin -int\, C(x)\}$$

for all $y \in K$. Then F is a KKM-mapping on K. In fact, suppose that F is not a KKM-mapping, then there exists $\{x_1, x_2, \ldots, x_n\}$ in K, $\sum_{i=1}^{n} \alpha_i = 1, \alpha_i \geq 0, i = 1, 2, \ldots, n$ such that $\sum_{i=1}^{n} \alpha_i x_i \notin \bigcup_{j=1}^{n} F(x_j)$, for any $j = 1, 2, \ldots, n$. Thus

$$\left\langle T\left(\sum_{i=1}^{n} \alpha_i x_i\right), \theta\left(x_j, \sum_{i=1}^{n} \alpha_i x_i\right)\right\rangle + \eta\left(\sum_{i=1}^{n} \alpha_i x_i, x_j\right) \in -int\, C(x), \quad \ldots (3.3)$$

for $j = 1, 2, \ldots, n$. On the other hand, by convexity of the operator,

$$\left\{\left\langle T\left(\sum_{i=1}^{n} \alpha_i x_i\right), \theta\left(\sum_{j=1}^{n} \beta_j x_j, \sum_{i=1}^{n} \alpha_i x_i\right)\right\rangle + \eta\left(\sum_{i=1}^{n} \alpha_i x_i, \sum_{j=1}^{n} \beta_j x_j\right)\right\}$$

$$\leq_{C(x)} \sum_{j=1}^{n} \left(\beta_j \left\{\left\langle T\left(\sum_{i=1}^{n} \alpha_i x_i\right), \theta\left(x_j, \sum_{i=1}^{n} \alpha_i x_i\right)\right\rangle + \eta\left(\sum_{i=1}^{n} \alpha_i x_i, x_j\right)\right\}\right),$$

where

$\sum_{j=1}^{m} \beta_j = 1, \beta_j \geq 0, j = 1, 2, \ldots, n$. So, we have

$$\sum_{j=1}^{n} \left(\beta_j \left\{\left\langle T\left(\sum_{i=1}^{n} \alpha_i x_i\right), \theta\left(x_j, \sum_{i=1}^{n} \alpha_i x_i\right)\right\rangle + \eta\left(\sum_{i=1}^{n} \alpha_i x_i, x_j\right)\right\}\right)$$

$$-\left\{\left\langle T\left(\sum_{i=1}^{n}\alpha_i x_i\right), \theta\left(x_j, \sum_{i=1}^{n}\beta_i x_j, \sum_{i=1}^{n}\alpha_i x_i\right)\right\rangle + \eta\left(\sum_{i=1}^{n}\alpha_i x_i, \sum_{j=1}\beta_j x_j\right)\right\} \in C(x).$$

Thus,

$$\left\langle T\left(\sum_{i=1}^{n}\alpha_i x_i\right), \theta\left(x_j, \sum_{i=1}^{n}\alpha_i x_i\right)\right\rangle + \eta\left(\sum_{i=1}^{n}\alpha_i x_i, x_j\right) \in C(x),$$

which contradicts to (3.3). Hence F is a KKM-mapping. Define another set-valued mapping $E : K \to 2^K$ by

$$E(y) := \{x \in K : \langle T(y), \theta(x, y)\rangle + \eta(y, x) \notin int\, C(x)\}$$

for all $y \in K$, then $F(y) \subset E(y)$. Indeed, if $x \in F(y)$, then $\langle T(x), \theta(y, x)\rangle + \eta(x, y) \notin -int\ C(x)$ for all $y \in K$. By the condition (3), we have $\langle T(y), \theta(x, y)\rangle + \eta(y, x) \notin int\, C(x)$ for all $y \in K$, thus $x \in E(y)$. Therefore, E is also a KKM-mapping on K. Now we claim that $E(y)$ is closed for all $y \in K$. In fact, let $\{x_n\}$ be a sequence in $E(y)$ such that $x_n \to x$. Since $x_n \in E(y)$, we have

$$\langle T(y), \theta(x_n, y)\rangle + \eta(y, x_n) \notin int\ C(x_n).$$

So,

$$-\{\langle T(y), \theta(x_n, y)\rangle + \eta(y, x_n)\} \in W(x_n).$$

Since W has a closed graph,

$$-\{\langle T(y), \theta(x, y)\rangle + \eta(y, x)\} \in W(x).$$

Thus,

$$\langle T(y), \theta(x, y)\rangle + \eta(y, x) \notin int\ C(x).$$

Hence, $x \in E(y)$, $E(y)$ is a closed subset of K. Since K is weakly compact, $E(y)$ is weakly compact for each $y \in K$. Thus by Theorem 2.1,

$$\bigcap_{y \in K} E(y) \neq 0.$$

i.e., there exists an $x_0 \in K$ such that $\langle T(y), \theta(x_0, y)\rangle + \eta(y, x_0) \notin int\ C(x_0)$, for all $y \in K$.

Moreover, if (4) and (5) hold, then by Theorem 3.1, there exists an $x_0 \in K$ such that $\langle T(x_0), \theta(y, x_0)\rangle + \eta(x_0, y) \notin -int\ C(x_0)$, for all $y \in K$.

Corollary 3.4 — When $L(X, Y) = X^*$, $Y = \mathbb{R}$, $C(x) = \mathbb{R}^+$, if η is a zero functional, then we obtain Theorem 5.1 in [2] as corollary.

REFERENCES

1. C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities, Applications to Free Boundary Problems*, John Wiley and Sons Ltd., New York, 1984.
2. A. Behera and G. K. Panda, *Indian J. pure appl. Math.* **28** (1997), 897-903.
3. G. Y. Chen, *J. Optim. Th. Appl.* **74** (1992), 445-56.
4. G. Y. Chen and X. Q. Yang, *J. math. Anal. Appl.* **153** (1990), 136-58.
5. K. Fan, *Math. Ann.* **142** (1961), 305-10.
6. F. Giannessi, Theorems of alternative, quadratic programmes and complementarity problems, In: *Variational Inequalities and Complementarity Problems* (*Ed.* R. W. Cottle, F. Giannessi and J. L. Lions), John WIley and Sons Ltd., Chichester, 1980, p 151-86.
7. F. Giannessi, On Minty variational principle. In: *New Trends in Mathematical Programming*, Kluwer Academic Publishers, Dordrect, 1997.
8. P. Hartman and G. Stampacchia, *Acta Math.* **115** (1966), 271-310.
9. D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities*, Academic Press, New York, 1980.
10. I. V. Konnov and J. C. Yao, *J. math. Anal. Appl.* **206** (1997), 42-58.
11. D. Kuroiwa, *Appl. Math. Lett.* **9** (1996), 97-101.
12. B. S. Lee and G. M. Lee, *Appl. Math. Lett.*, **12** (1999), 43-50.
13. B. S. Lee, G. M. Lee and D. S. Kim, *J. Korean math. Soc.* **33** (1996), 609-24.
14. B. S. Lee, G. M. Lee and D. S. Kim, *Indian J. pure appl. Math.* **28** (1997), 33-41.
15. G. M. Lee, D. S. Kim and H. Kuk, *J. math. Anal. Appl.*, **220** (1998), 90-98.
16. G. M. Lee, D. S. Kim, B. S. Lee and S. J. Cho, *Appl. Math. Lett.* **6** (1993), 47-51.
17. G. M. Lee, B. S. Lee and S. S. Chang, *J. math. Anal. Appl.* **203** (1996), 626-38.
18. G. Minty, *Duke Math. J.* **29** (1962), 341-46.
19. S. B. Nadler, Jr., *Pac. J. Math.*, **30** (1969), 475-88.
20. S. Park, B. S. Lee and G. M. Lee, *Int. J. Math. & math. Sci.* **21** (1998), 637-42.
21. A. H. Siddiqi, Q. H. Ansari and A. Khaliq, *J. Optim. Th. Appl.* **84** (1995), 171-80.
22. S. J. Yu and J. C. Yao, *J. Optim. Th. Appl.* **89** (1996), 749-69.