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STOCHASTIC INTEGRAL OF L_ϕ FUNCTION AND APPROXIMATION TO RFS SERIES

C. NAYAK*, I. P. MOHANTY** AND R. MISHRA***

**Department of Mathematics, G M College, Sambalpur, Orissa, India*
***Govt. College, Rourkela*
****Sambalpur University, Jyotivihar*

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In this paper the existence of the stochastic integral $\int_a^b f(t)\, g(t)\, dx\,(t, \omega)$ has been established, where f and g are functions in $L_\phi\,[a, b]$ and $L_\psi[a, b]$, (ϕ, ψ) being a complementary pair in the sense of Young and $x(t, \omega)$ is a symmetric stable process of index $\alpha, 0 < \alpha < 1$. Also the convergence in $(C, 1)$ probability of the Random Fourier-Stieltjes series $\sum_{-\infty}^{\infty} a_k\, b_k\, A_k\,(\omega)\, e^{2\pi k i y}$, where a_k and b_k are Fourier coefficient of f and g in L_ϕ and L_ψ respectively has been discussed.

Key Words : RFS Series - Random Fourier Stieltjes Series

1. INTRODUCTION

Let $x(t, \omega)$ be a continuous homogenous stochastic process with independent increments. It is well known that (cf. Lukacs)[2] the stochastic integral

$$\int_a^b f(t)\, dx\,(t, \omega) \qquad \dots (1)$$

exists in the sense of convergence in quadratic mean if $f \in L^2\,[a, b]$ and in the sense of convergence in probability if $f \in C\,[a, b]$. Recently, Nayak, Pattanayak and Mishra[4] have extended it to the case when $f \in L^p\,[a, b], p > 1$ and $x(t, \omega)$ a symmetric stable process of index $\alpha, 0 < \alpha \leq 2, \alpha \leq p$. Defining the random variable $A_k\,(\omega)$ as

$$A_k\,(\omega) = \int_0^1 e^{-2\pi k i t}\, dx\,(t, \omega),$$

which is the Fourier-Stieltjes coefficient of the process $x(t, \omega)$, they have also shown that the weighted random Fourier-Stieltjes series (RFS series in short)

$$\sum_{-\infty}^{\infty} a_k A_k(\omega) e^{2\pi k i y}, \quad \text{... (2)}$$

converges in probability to the stochastic integral

$$\int_0^1 f(y-t)\, dx(t, \omega), \quad \text{... (3)}$$

where a_k is the Fourier coefficient of $f \in L^p[0, 1], p > 1$.

Very recently Das, Ellipse and Pattanayak[1] have further extended the results of[4]. They have shown the existence of the stochastic integral (1) in the sense of convergence in probability if (*a*) $f \in L_\phi[a, b]$ and $x(t, w)$ a continuous, homogenous symmetric process or (*b*) $f \in L^p[a, b], 0 < p < \infty$, and $x(t, \omega)$ a symmetric stable process of index $\alpha, 0 < \alpha \leq 2,\ \alpha \leq p$. But in all above results they have considered the function $\phi = -\log f(u, \theta)$, where $f(u, \theta)$ is the characteristic function of the increment $x(t + \theta, \omega) - x(t, \omega)$.

In the present paper we have established the existence of the stochastic integral

$$\int_a^b f(t)\, g(t)\, dx(t, \omega), \quad \text{... (4)}$$

where f and g are functions in $L_\phi[a, b]$ and $L_\Psi[a, b]$, (ϕ, Ψ) being a complementary pair in the sense of Young and $x(t, \omega)$ is a symmetric stable process of index $\alpha, 0 < \alpha < 1$. We have also discussed the convergence in $(C, 1)$ probability of the double weighted RFS series.

$$\sum_{-\infty}^{\infty} a_k b_k A_k(\omega) e^{2\pi k i y}, \quad \text{... (5)}$$

where the weights a_k and b_k are Fourier coefficients of $f \in L_\phi[0, 1]$ and $g \in L_\Psi[0, 1]$ respectively.

For smooth understanding, in section 2, we have elaborated in brief about the complementary pair of functions in the sense of Young. Moreover, our approach differs from [1] in the sense that everything is discussed through the norm defined in L_ϕ space.

2. Complementary Functions in the Sense of Young and L_ϕ Norm

The class of function 'f' such that $\phi(|f|)$ is integrable over (a, b) is called the class $L_\phi[a, b]$. When $\phi(u) = u^p$, we get the class of function 'f' such that $|f|^p$ is integrable over (a, b), i.e., the class $L^p[a, b]$. Thus, the class $L_p[a, b]$ is a particular class of $L_\phi[a, b]$. When 'f' is in $L^p[a, b]$ the norm called L_p norm is defined as

$$\|f\|_p = \left(\int_a^b |f|^p \right)^{\frac{1}{p}}.$$

Next, we define a norm in $L_\phi[a, b]$ with the help of complementary functions Ψ and ϕ in the sense of Young.

Let $\phi(u), u \geq 0$ amd $\psi(v), v \geq 0$ be two functions which are continuous, vanishing at origin, strictly increasing and inverse of each other. Then for $a, b \geq 0$ we get the inequality

$$a\, b \leq \phi(a) + \psi(b), \qquad \text{... (6)}$$

where

$$\phi(x) = \int_0^x \phi(u)\, du$$

and

$$\Psi(y) = \int_0^y \psi(v)\, dv.$$

The functions ϕ and Ψ are called complementary functions in the sense of Young and inequality (6) is called Young's inequality. The equality holds, when $b = \phi(a)$. A simple example of this can be seen, when $\phi(u) = u^\alpha$ and $\psi(v) = v^{1/\alpha}$, where $\alpha > 0$. It can be noted that every function $\phi(u), u \geq 0$, which is non-negative, convex, vanishing at origin and for which $\frac{\phi(u)}{u} \to \infty$ with 'u' is a Young function and for such a function, there corresponds a function ψ of same type such that the pair ϕ and ψ satisfy Young's inequality (cf. [6] Zygmund P. 16). By the help of function $g \in L_\psi[a, b]$ one can define a norm in $L_\phi[a, b]$ as

$$\|f\|_\phi = \sup_g \left| \int_a^b f(t)\, g(t)\, dt \right|,$$

the supremum being taken over all $g \in L_\psi[a, b]$ such that $\rho_g = \int_a^b \Psi(|g|)\, dt \leq 1$. When $\|f_\phi\|$ is finite the class is denoted by $L^*_\phi[a, b]$ which obviously contains $L_\phi[a, b]$. Thus, $L_\phi \subset L^*_\phi$ and L^*_ϕ is a complete normed space (cf [6] Zygmund).

Hence,

$$\|f_n - f_m\|_\phi \to 0 \text{ for } m, n \to \infty \Rightarrow \|f - f_n\|_\phi \to 0 \text{ for } n \to \infty$$

for some f belonging to L_ϕ.

One can also define a norm in L_ϕ in an alternative way as the lower bound of all λ's satisfying

$$\int_a^b \phi(\lambda^{-1} |f|)\, dt \le \phi(1),$$

which is non-negative and is zero if and only if $f \equiv 0$. This norm is denoted as $N_\phi f$ and it can be checked that

$$N_\phi(\alpha f) = |\alpha| N_\phi f$$

and

$$N_\phi(f+g) \le N_\phi f + N_\phi g.$$

The norms $N_\phi f$ and $\|f\|_\phi$ are equivalent in the sense that their ratio is bounded above and below by 2 and $\phi(1)$, i.e.,

$$\phi(1) \le \frac{\|f\|_\phi}{N_\phi f} \le 2 \text{ (cf. [6] Zygmund p. 175).}$$

With respect to the norm $N_\phi f$ the general Holder's inequality for the complementary pair (ϕ, Ψ) holds good giving the result

$$\left| \int_a^b f \cdot g dt \right| \le N_\phi f \cdot N_\Psi g,$$

where $f \in L_\phi^*(a, b)$ and $g \in L_\Psi^*(a, b)$

A sequence of function f_n in $L_\phi(a, b)$ is said to converge to 'f' in $\|\cdot\|_\phi$ norm if $\|f_n - f\|_\phi \to 0$ as $n \to \infty$, i.e.,

$$\sup_g \left| \int_a^b (f_n - f)\, g(t)\, dt \right| \to 0 \text{ as } n \to \infty$$

supremum being taken over all g in $L_\Psi(a, b)$ with

$$\rho_g = \int_a^b \Psi(|g|)\, dt \le 1$$

3. Results

***Theorem* 1** — *Let $x(t, \omega)$ be a symmetric stable process with characteristic function $e^{-c|t|^\alpha}$, $0 < \alpha < 1$, periodic with period one almost surely. Let $f(t) \in L_\phi[a, b]$ and $g(t) \in L_\Psi[a, b]$,* *where ϕ and Ψ are complementary pair in the sense of Young. Then the stochastic integral*

$$\int_a^b f(t)\, g(t)\, dx\,(t, w),$$

exists in the sense of convergence in probability.

In the next theorem we have approximated the above stochastic integral to a random Fourier-Stieltjes series.

Theorem 2 — *Let $x\,(t, \omega)$ be as in Theorem 1. Let $A_k\,(\omega)$ be Fourier-Stieltjes coefficient of $x(t, \omega)$ and a_k, b_k be Fourier coefficients of $f \in L_\phi$ (0, 1) and $g \in L_\Psi(0, 1)$ respectively, where ϕ, Ψ are complementary in the sense of Young, then the RFS series*

$$\sum_{-\infty}^{+\infty} a_k\, b_k\, A_k\,(\omega)\, e^{2\pi k i y}$$

converges in (C, 1) probability to the stochastic integral $\int_0^1 h(y-t)\, dx\,(t, w)$, $h(x)$ being convolution of 'f' and 'g'.

For the proof of the theorem - 1 we require the following lemmas :

Lemma 1 — For any function $f \in L_\phi\,[a, b]$ there exists a sequence of continuous functions $H_n\,(t)$ which converges to 'f' in the norm $\|\;\|_\phi$.

Lemma 2 (cf. [5] Samal and Mishra) — Let $f(t)$ be continuous function in $[a, b]$ and $x(t)$ a symmetric stable process with characteristic function $e^{-c|t|^\alpha}, 0 < \alpha \le 2$, then for $\delta > 0$

$$p\left(\left| \int_a^b f(t)\, dx\,(t) \right| > \delta \right) \le \frac{C \cdot 2^{\alpha+1}}{(1+\alpha)\,\delta^\alpha} \int_a^b |f(t)|^\alpha\, dt$$

C being a positive constant.

PROOF OF LEMMA 1 — For every large N define function 'h' as

$$h(t) = \begin{cases} f(t), \text{ for } f(t) < N \\ 0, \text{ elsewhere} \end{cases}$$

Thus $h(t)$ is a bounded function and choosing N large enough $f(t) = h(t)$ except a set of measure zero. Hence,

$$\int_a^b |f-h|\, dt < \frac{\varepsilon}{k}, \text{ where } k = 2\,\beta\,(b-a)$$

(β is to be fixed later on according to convenience).

By definition of $\| \|_\phi$ norm

$$\| f-h \|_\phi = \sup_g \left| \int_a^b (f-h)\, g\, dt \right| ,$$

where the sup is taken over all g satisfying

$$\rho_g = \int_a^b \Psi(|g|)\, dt \le 1.$$

Now choosing β such that $(b-a)\,\Psi(\beta) = 1$ and choosing

$$g(t) = \beta\, sign\, (f-h)$$

we get

$$\int_a^b \Psi(|g|)\, dt = \int_a^b \Psi(\beta)\, dt \le 1$$

and

$$\| f-h \|_\phi = \sup_g \int_a^b |f-h|\,|g|\, dt \le \frac{\varepsilon}{k}\beta\,(b-a) = \frac{\varepsilon}{2}. \qquad \text{... (7)}$$

Now defining functions $H(x)$ and $h_n(x)$ as

$$H(x) = \int_0^x h(t)\, dt$$

and

$$h_n(x) = n\,[H(x+1/n) - H(x)]$$

$$= \frac{H\left(x+\frac{1}{n}\right) - H(x)}{1/n} = \frac{\int_0^{x+\frac{1}{n}} h(t)\, dt - \int_0^x h(t)\, dt}{1/n} = \int_x^{x+\frac{1}{n}} \frac{h(t)}{1/n}\, dt.$$

Thus, for n large enough $h_n(x)$ being integrals are absolutely continuous and tend to $h(x)$, by Lebesgue theorem on differentiation. So we get

$$|h(x) - h_n(x)| < \frac{\varepsilon}{k} \text{ where } k = 2\,\beta\,(b-a).$$

Also

$$\| h - h_n \|_\phi = \sup_g \left| \int_a^b (h - h_n)\, g(t)\, dt \right|$$

$$\leq \sup_g \int_a^b | h - h_n | | g | \, dt < \frac{\varepsilon}{2} \text{ (as in 7).} \qquad \ldots (8)$$

Now from (7) and (8) we get

$$\| f - h_n \| \leq \| f - h \|_\phi + \| h - h_n \|_\phi < \varepsilon$$

So that

$$\| f - h_n \| \to 0 \text{ as } n \to \infty.$$

which proves Lemma 1.

Proof of Theorem 1

By Lemma (1) for every $f \in L_\phi\ [a, b]$ there is a sequence of continuous function f_m which converges to f in L_ϕ norm. Thus,

$$\| f - f_m \|_\phi \to 0 \text{ as } m \to \infty.$$

Also L_ϕ being complete normed space

$$\| f_n - f_m \| \to 0 \text{ as } m, n \to \infty.$$

Considering the sequence of stochastic integral

$$\int_a^b f_m(t)\, g(t)\, dx(t),$$

we get
$$p\left(\left| \int_a^b f_m(t)\, g(t)\, dx(t) - \int_a^b f_n(t)\, g(t)\, dx(t) \right| > \delta \right)$$

$$= p\left(\left| \int_a^b (f_m - f_n)\, g(t)\, dx(t) \right| > \delta \right)$$

$$\leq \mu \int_a^b | f_m f_n |^\alpha | g |^\alpha dt, \text{ (μ is a constant) (by lemma 2)}$$

$$\leq \mu \left(\int_a^b (|f_m - f_n|^{\alpha} |g|^{\alpha})^{\frac{1}{\alpha}} \right)^{\alpha} \left(\int_a^b dt \right)^{\frac{1}{1-\alpha}}, 0 < \alpha < 1 \qquad \text{(by Holder's Inequality)}$$

$$= \mu \left(\int_a^b |f_m - f_n| \, |g| \, dt \right)^{\alpha} (b-a)^{\frac{1}{1-\alpha}}$$

$\leq \mu N_{\phi}(f_m - f_n) N_{\Psi}(g)$ (by general Holder's inequality. (μ being a constant)

$\leq \mu N_{\phi}(f_m - f_n)$, since $N_{\Psi}(g) \leq 1$

$\sim \|f_m - f_n\|_{\phi} \to 0$ (for $N_{\phi} f \approx \|f\|_{\phi}$).

Thus the sequence of stochastic integral

$$\int_a^b f_m(t) \, g(t) \, dx(t).$$

converges in probability to a random variable Y which we define as

$$\int_a^b f(t) \, g(t) \, dx(t).$$

That it is independent of the choice of sequence can be shown as follows : Suppose there is a sequence g_m such that

$$\|f - g_m\|_{\phi} \to 0.$$

Now

$$\lim_{m \to \infty} \|f_m - g_m\|_{\phi} \leq \|f_m - f\|_{\phi} + \|g_m - f\|_{\phi} \to 0.$$

Thus, we have

$$p\left(\left| \int_a^b f_m(t) \, g(t) \, dx(t) - \int_a^b g_m(t) \, g(t) \, dx(t) \right| > \delta \right)$$

$$= p\left(\left| \int_a^b (f_m - g_m) \, g(t0 \, dx(t) \right| > \delta \right)$$

$$\leq \mu \int_a^b |f_m - g_m|^{\alpha} |g(t)|^{\alpha} dt, \text{ (by Lemma 2)}$$

$$\le \left(\int_a^b (|(f_m - g_m)\, g|^{\alpha})^{1/\alpha} \right)^{\alpha} (b-a)^{\frac{1}{1-\alpha}}, 0 < \alpha < 1$$

(by Holder's inequality)

$$\le \mu N_\phi (f_m - g_m) N_\Psi (g)$$

$$\le \mu N_\phi (f_m - g_m) \text{ (for } N_\Psi g \le 1)$$

$$\approx \mu \| f_m - g_m \|_\phi \to 0$$

PROOF OF THEOREM 2 — Let $S_n(y, \omega) = \sum_{-n}^{+n} a_k b_k A_k(\omega) e^{2\pi k i y}$,

$$\delta_n'(y, \omega) = \frac{S_0 + \ldots\ldots + S_{n-1}}{n},$$

$$\$_n(y) = \sum_{-n}^{n} a_k b_k e^{2\pi k i y},$$

and

$$\delta_n(y) = \frac{\$_0 + \ldots. + \$_{n-1}}{n}$$

Thus, we obtain

$$S_n(y, \omega) = \sum_{-n}^{n} a_k b_k A_k(\omega) e^{2\pi k i y}$$

$$= \int_0^1 \sum_{-n}^{n} a_k b_k e^{2\pi k i(y-t)} dx(t, w) = \int_0^1 S_n(y-t)\, dx(t, w),$$

Similarly, we have

$$\delta_n'(y, \omega) = \int_0^1 \delta_n(y-t)\, dx(t, w),$$

Now we see that

$$p(|\delta_n'(y, \omega) - \delta_m'(y, \omega)| > \delta)$$

$$= p\left(\int_0^1 |(\delta_n - \delta_m)\, dx\,(t, w)| > \delta \right)$$

$$\leq \mu \int_0^1 |\delta_n - \delta_m|^{\alpha}\, dt$$

(by Lemma 2)

$$\leq \mu \left(\int_0^1 (|\delta_n - \delta_m|^{\alpha})^{1/\alpha} \right)^{\alpha}$$

(by Holder's inequality)

$$= \mu \left(\int_0^1 |\delta_n - \delta_m|\, dt \right)^{\alpha}. \qquad \dots (9)$$

It is known that (cf. Zygmund[6] p. 178 Vol. 1) if a_k and b_k are Fourier coefficients of $f \in L_\phi$ and $g \in L_\Psi$ respectively, then $(a_k b_k)$ sare Fourier coefficient of the convolution $h(x)$ of f and g and is summable (C.1).

Thus, we have

$$\int_a^b |\delta_n - \delta_m|\, dt \to 0 \text{ as } m, n \to \infty$$

and

$$\int_a^b |\delta_n - h|\, dt \to 0 \text{ as } n \to \infty. \qquad \dots (10)$$

From (9) we conclude that the stochastic integral $\int_0^1 \delta_n\, dx(t)$ converges in probability, thus

$$= p\left(\left| \delta_n'((y-t), \omega) - \int_0^1 h(y-t)\, dx\,(t) \right| > \delta \right)$$

$$\leq \mu \int_0^1 |\delta_n(y-t) - h(y-t)|^{\alpha}\, dt \text{ (by Lemma 2)}$$

$$\leq \mu \left(\int_0^1 |\delta_n(y-t) - h(y-t)|^\alpha \right)^{\frac{1}{\alpha}} \left(\int_0^1 dt \right)^{\frac{1}{1-\alpha}}$$

(by Holder's inequality)

$$\leq \mu' \left(\int_0^1 |\delta_n - h| \, dt \right)^\alpha \to 0 \text{ (from 10)}$$

which proves the theorem.

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