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WHICH MILNOR MANIFOLDS BOUND?

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The object of this paper is to determine those Milnor manifolds which bound.

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1. INTRODUCTION

Milnor manifolds were first introduced by Milnor[3]. For positive integers m and n, the Milnor manifold $H(m, n)$ is the $m + n - 1$ dimensional submanifold of $\mathbf{R}P^m \times \mathbf{R}P^n$ given by

$$\left\{ ([x_0, \ldots, x_m], [y_0, \ldots, y_n]) \in \mathbf{R}P^m \times \mathbf{R}P^n : \sum_{i=0}^{\min(m,n)} x_i y_i = 0 \right\}$$

In fact $H(m, n)$ is the submanifold of $\mathbf{R}P^m \times \mathbf{R}P^n$ dual to $(a + b)$; a and b being the generators of $H^*(\mathbf{R}P^m; Z_2)$ and $H^*(Rp^n; Z_2)$ respectively. Note that $a^{m+1} = b^{n+1} = 0$, whereas a^i and b^j are nonzero for $0 \le i \le m$ and $0 \le j \le n$. The total Stiefel-Whitney class $W(H(m, n))$ is given by the restriction to $H(m, n)$ of

$$\frac{(1+a)^{m+1}(1+b)^{n+1}}{(1+a+b)} \qquad \ldots (*)$$

and the Stiefel-Whitney number of $H(m, n)$ corresponding to a partition $i_1 + i_2 + \ldots + i_k = m + n - 1$ is given by

$$\langle (a+b) W_{i_1} \ldots W_{i_k}, [\mathbf{R}P^m \times \mathbf{R}P^n] \rangle \in Z_2,$$

where W_{i_j} is the sum of i_j-dimensional terms in the expansion of the expression (*) mentioned above. More specifically

$$W_{ij} = \sum_{t=0}^{i_j} \Sigma \binom{m+1}{f}\binom{n+1}{g}\binom{i+1}{j} a^t b^{i_j - t},$$

where the second summation is over all f, g, i and j such that $f + i = t$ and $g + j = i_j - t$, and the symbols ( ) stand for the binomial coefficients reduced modulo 2. Thus $H(m,n)$ does not bound if and only if

$$(a+b)\, W_{i_1} \ldots W_{i_k} = a^m b^n,$$

for some partition $i_1 + \ldots + i_k = m + n - 1$. It is also well known that $H(m, n)$ fibres over $\boldsymbol{R}P^m$ with fibre $\boldsymbol{R}P^{n-1}$, if $m \le n$. One may note here that $H(m, n) \cong H(n, m)$. Milnor manifolds give generators of unoriented bordism algebra. But practically nothing is known about the bounding of Milnor manifolds. The corresponding problem for Dold manifolds has been settled in [2]. In this paper, we consider the question "Which Milnor manifolds bound?" and answer it as follows :

Theorem 1.1 — *A Milnor manifold $H(m, n)$ with $m \le n$, bounds if and only if at least one of the following conditions holds :*

(a) $m = n$,

(b) $m = 1$,

(c) $mn \equiv 1 \pmod 2$ *and*

(d) $n \equiv 2 \pmod 4$ and $m + 1 < 2^{\nu(n+2)}$, where $\nu(n+2)$ is the largest integer such that $2^{\nu(n+2)} \mid (n+2)$.

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2. BOUNDING OF $H(m, n)$ WHEN $m = n$ OR m IS ODD

We first consider few simple cases.

Proposition 2.1 — $H(m, m)$ bounds.

PROOF : It is enough to note that

$$T([x_0, \ldots, x_m], [y_0, \ldots, y_m]) = ([y_0, \ldots, y_m], [x_0, \ldots, x_m])$$

defines a fixed point free involution on $H(m, m)$. □

Proposition 2.2 — $H(2r+1, 2q+1)$ bounds.

PROOF : It is enough to note that

$$T([x_0, \ldots, x_{2r+1}], [y_0, \ldots, y_{2q+1}])$$

$$= ([-x_1, x_0, \ldots, -x_{2r+1}, x_{2r}], [-y_1, y_0, \ldots, -y_{2q+1}, y_{2q}])$$

defines a fixed point free involution on $H(2r+1, 2q+1)$. □

Proposition 2.3 — $H(1, n)$ bounds.

PROOF : Comparing the Stiefel-Whitney classes and hence the Stiefel-Whitney numbers, one can see that $H(1, n)$ is bordant to $\boldsymbol{R}P(\xi \oplus (n-1)\boldsymbol{R})$, where $\xi \oplus (n-1)\boldsymbol{R} \to \boldsymbol{R}P^1$ is the Whitney sum of the canonical line bundle and the trivial $(n-1)$-plane bundle over $\boldsymbol{R}P^1$. But, by 2.2 of [1], $\boldsymbol{R}P(\xi \oplus (n-1)\boldsymbol{R})$ bounds. Hence $H(1, n)$ bounds. □

Proposition 2.4 — $H(2r+1, 2^\beta(2B+1))$ does not bound if at least one of the following holds :

(i) $\beta \geq 1$ and $2r+1 > 2^\beta(2B+1)$ and

(ii) $\beta > 1$ and $r > 0$.

PROOF : Suppose (*i*) holds. Then $H(2r+1, 2^\beta(2B+1))$ fibres over $\boldsymbol{R}P^{2^\beta(2B+1)}$ with fibre $\boldsymbol{R}P^{2r}$. Therefore,

$$\chi(H(2r+1, 2^\beta(2B+1)) = \chi(\boldsymbol{R}P^{2^\beta(2B+1)})\,\chi(\boldsymbol{R}P^{2r}) \neq 0.$$

Here χ stands for the Euler characteristic reduced modulo 2.

Thus $H(2r+1, 2^\beta(2B+1))$ does not bound, if (*i*) holds.

On the other hand, suppose (*ii*) holds. Then the total Stiefel-Whitney class $W(H(2r+1, 2^\beta(2B+1)))$ is given by

$$\left(1+\sum_{i=0}^{\infty} ab\,(a+b)^i\right)(1=a)^{2r+1}\,(1+b)^{2^\beta(2B+1)}.$$

Therefore,

$$W_1 = a,$$

$$W_3 = \begin{cases} ab^2, & \text{if } r \text{ is even} \\ a(a+b)^2, & \text{ifr is odd.} \end{cases}$$

$W_{2^\beta-4} = ab\,(a+b)^{2^\beta-6}$ + terms divisible by a^2, if $\beta > 2$ and $W_{2\beta} = b^{2^\beta}$ + terms divisible by a.

Hence,

$$(a+b)\,W_1^{2r-2}\,W_3^2\,W_{2^\beta-4}\,W_{2\beta}^{2B} = a^{2r+1}\,b^{2^\beta(2B+1)}, \text{ if } \beta > 2,$$

and

$$(a+b)\,W_1^{2r-2}\,W_3^2\,W_{2\beta}^{-4}\,W_{2^\beta}^{2B} = a^{2r+1}\,b^{2^\beta(2B+1)}, \text{ if } \beta = 2.$$

Thus $H(2r+1, 2^\beta(2B+1))$ does not bound, if (ii) holds. □

In order to investigate the remaining cases of H (odd, even), we first establish the following:

Lemma 2.5 — If $B+1=2^{\theta}$ odd, then

$$(a+b)\, W_1^i\, W_2^j = a^{2r+1}\, b^{2(2B+1)}$$

for some i, j if and only if $r \geq 2^{\theta+1}-1$; W_1 and W_2 being the first and second Stiefel-Whitney classes of $H(2r+1, 2(2B+1))$.

PROOF : We have

$$W_1 = a$$

and

$$W_2 = \begin{cases} b(a+b), & \text{if } r \text{ is even} \\ ab+(a+b)^2, & \text{if } r \text{ is odd.} \end{cases}$$

Now, suppose $r \geq 2^{\theta+1}-1$. Consider $i = 2r-2^{\theta+2}+2$ and $j = 2B+2^{\theta+1}$. If r is even, the coefficient of $a^{2r+1}\, b^{2(2B+1)}$ in $(a+b)\, W_1^i\, W_2^j$ is

$$\binom{2(B+1)+2^{\theta+1}-1}{2^{\theta+2}-1} = \binom{2^{\theta+1}\cdot odd + 2^{\theta+1}-1}{2^{\theta+2}-1} \neq 0.$$

If r is odd, the coefficient of $a^{2r+1}\, b^{2(2B+1)}$ is

$$\sum_k \binom{2(B+1)+2^{\theta+1}-2}{k} = \binom{2(2^{\theta+1}\cdot odd + 2^{\theta+2}-2-k)+1}{2^{\theta+2}-1-k}$$

which is again nonzero, since the first factor is nonzero only when k is even; and $2^{\theta+2}-2$ is the only even value of k for which the second factor is nonzero.

Conversely, suppose $r < 2^{\theta+1}-1$. We shall show that there do not exist any i, j for which

$$(a+b)\, W_1^i\, W_2^j = a^{2r+1}\, b^{2(2B+1)}$$

From dimensional considerations, it is clear that i cannot be odd. Also, for each i, j, we have

$$(a+b)\, a^{2i}\, (ab+(a+b)^2)^j = \sum_{k=0}^{j} \binom{j}{k} (a+b)\, a^{2(i+k)}\, (b(a+b))^{j-k}.$$

Thus, it is enough to show that, for each i, j, the coefficient, say C, of $a^{2r+1}\, b^{2(2B+1)}$ in $(a+b)\, a^{2i}\, (b(a+b))^j$ is zero. Now,

$$C = \binom{2(B+1)+r-i}{1+2(r-i)} = \binom{r-i+2^{\theta+1}\cdot odd}{1+2\,(r-i)}.$$

Further, $r < 2^{\theta+1} - 1 \Rightarrow r - i < 2^{\theta+1} - 1$. Let ϕ be the smallest non-negative integer such that

$$\binom{r-i}{2^{\phi}} = 0.$$

Clearly, $\phi \leq \theta$. Therefore, $C = 0$, since

$$\binom{2(r-i)}{2^{\phi}} = 1.$$

This completes the proof. □

Theorem 2.6 — *If $B + 1 = 2^{\theta}$.odd then $H(2r + 1, 2(2B + 1))$ does not bound if and only if $r \geq 2^{\theta+1} - 1$.*

PROOF : If $r \geq 2^{\theta+1} - 1$ then, by Lemma 2.5, $H(2r + 1, 2(2B + 1))$ does not bound. Conversely, suppose $r < 2^{\theta+1} - 1$. Now, the total Stiefel-Whitney class

$$W(H(2r+1, 2(2B+1))) = \frac{(1+W_1)^{2r+1}(1+b^{2^{\theta+2}})^{odd}}{1+W_1+W_2+\delta W_1^2},$$

where $\delta = 0$ or 1, according as r is even or odd. Thus, W is a function of W_1, W_2 and $b^{2^{\theta+2}}$. So, to complete the proof of theorem, it is enough to show that there do not exist any i, j for which

$$(a+b)\, W_1^i\, W_2^j\, (b^{2^{\theta+2}})^k = a^{2r+1}\, b^{2(2B+1)}.$$

But this follows immediately, once we observe that any such i, j would also satisfy the equation

$$(a+b)\, W_1^i\, W_2^j = a^{2r+1}\, b^{2(2B'+1)}$$

for the Milnor manifold $H(2r+1, 2(2B'+1))$, where $B'+1 = B - 2^{\theta}\cdot k + 1 = 2^{\theta'}$ odd for some $\theta' \geq \theta$, which is a contradiction to Lemma 2.5. □

3. BOUNDING OF $H(m, n)$, WHEN $m \neq n$ AND BOTH m, n ARE EVEN

In this section we shall discuss the bounding of $H(2^{\alpha}(2A+1), 2^{\beta}(2B+1))$, where $\alpha, \beta \geq 1$. Since $H(m, n) \approx H(n, m)$, we assume that $2^{\alpha}(2A+1) < 2^{\beta}(2B+1)$. First we consider the case $\beta > 1$.

Theorem 3.1 — *$H(2^{\alpha}(2A+1), 2^{\beta}(2B+1))$ does not bound, if $\beta > 1$.*

PROOF : Here the total Stiefel-Whitney class $W(H(2^{\alpha}(2A+1), 2^{\beta}(2B+1)))$ is given by

$$\left(1 + \sum_{i=0}^{\infty} ab\,(a+b)^i\right)(1+a^{2^{\alpha}})^{2A+1}(1+b^{2^{\beta}})^{2B+1}.$$

Therefore,

$$W_2 = \begin{cases} ab, & \text{if } \alpha > 1 \\ a(a+b), & \text{if } \alpha = 1 \end{cases}$$

$$W_3 = ab\,(a+b)$$

and $W_{2^\beta} = b^{2^\beta}$ + terms divisible by a.

Case I — $\alpha < \beta$.

In this case $2^\beta(2B+1) - 2^\alpha(2A+1) = 2^\beta \cdot t + 2s$ where $t \geq 0$ and $0 < 2s < 2^\beta$, and we have

$$(a+b)\, W_2^{(2^\alpha(2A+1)-2)}\, W_3\, (W_{2\beta})^t\, W_{2s} = a^{2^\alpha(2A+1)}\, b^{2^\beta}\, (2B+1),$$

noting that $W_{2s} = ab\,(a+b)^{2s-2}$ + terms divisible by a^2.

Case II — $\alpha > \beta$.

In this case $W_2 = ab$ and we have

$$(a+b)\, W_2^{2^\alpha(2A+1)-2^\beta+1}\, W_3^{2^\beta-1}\, W_{2\beta}^{2B-2^{\alpha-\beta}(2A+1)} = a^{2^\alpha(2A+1)}\, b^{2^\beta(2B+1)}.$$

Case III — $\alpha = \beta$.

In this case we have

$$(a+b)\, W_2^{2^\alpha \;\; 2A+1}\, W_3^{2\alpha-1}\, W_{2\beta}^{2(B-A)-1} = a^{2^\alpha(2A+1)}\, b^{2^{\beta(2B+1)}}$$

Hence it follows that $H(2^\alpha(2A+1), 2^\beta(2B+1))$ does not bound if $\beta > 1$. □

We now consider the case $\beta = 1$. In fact we shall settle this case part by part.

Theorem 3.2 — *Let* $B+1 = 2^\theta \cdot odd$. *Then* $H(2^\alpha(2A+1), 2\,(2B+1))$ *bounds if* $2^\alpha(2A+1) < 2^{\theta+2}$.

PROOF : In this case, $W_2 = b(a+b) + \delta \cdot a^2$, where $\delta = 0$ or 1 according as $\alpha > 1$ or $\alpha = 1$, $W_3 = ab\,(a+b)$, and the total Stiefel-Whitney class $W(H(2^\alpha(2A+1), 2(2B+1)))$ is given by

$$\frac{(1+a)^{2^\alpha(2A+1)+2}\,(1+b)^{2(2B+1)+2}}{(1+a)\,(1+b)\,(1+a+b)} = \frac{(1+a^2)^{2^{\alpha-1}(2A+1)+1}\,(1+b^{2^{\theta+2}})^{odd}}{1+W_3+W_2+(1-\delta)\,a^2}.$$

Thus, W is a function of W_2, W_3, a^2 and $b^{2^{\theta+2}}$. We shall show that there do not exist any i, j, k, t such that

$$(a+b)\, a^{2i}\, b^{2^{\theta+2}\cdot k}\, W_2^j\, W_3^t = a^{2^\alpha(2A+1)}\, b^{2(2B+1)}.$$

Clearly, by dimensional considerations t can not be even. Also, for all i, j, k, t, we have

$$(a+b)\,a^{2i}\,b^{2^{\theta+2}\cdot k}\,(a^2+b(a+b))^j\,(ab\,(a+b))^t$$

$$=\sum_{p=0}^{j}\binom{j}{p}(a+b)\,a^{2(i+p)}\,(b^{2^{\theta+2}\cdot k}\,(b(a+b))^{j-p}\,(ab\,(a+b))^t.$$

Hence, it is enough to show that the coefficient, say G of $a^{2^{\alpha}(2A+1)}\,b^{2(2B+1)}$ in

$$(a+b)\,a^{2i}\,b^{2^{\theta+2}\cdot k}\,(b\,(a+b))^j\,(ab\,(a+b))^t$$

is zero for all i, j, k, t with t odd. Now, $G=\binom{I}{J}$, where

$$I=\frac{2^{\theta+2}\cdot D-1+(2^{\alpha}(2A+1)-2i-t)}{2}$$

$$J=2^{\alpha}(2A+1)-2i-t.$$

If $D=0$ then $I<J$, so that $G=0$. So, let $D\neq 0$. Let λ be the smallest non-negative integer such that $\binom{J}{2^{\lambda}}=0$. Since t is odd and $2^{\alpha}(2A+1)<2^{\theta+2}$, we have $2\leq 2^{\lambda}\leq 2^{\theta+1}$. So, $\binom{I}{2^{\lambda-1}}=0$ whereas $\binom{J}{2^{\lambda-1}}=1$. Therefore, once again we have $G=0$. Hence the theorem follows. □

***Theorem* 3.3** — *Let $B+1=2^{\theta}\cdot$ odd. Then $H(2^{\alpha}(2A+1),2(2B+1))$ does not bound if $\alpha>1$ and $2^{\alpha}(2A+1)\geq 2^{\theta+2}$.*

PROOF : In this case $W_2=b(a+b)$ and $W_3=ab\,(a+b)$. We shall show that there exist positive integers I and J such that

$$(a+b)\,W_2^{I}\,W_3^{J}=a^{2^{\alpha}(2A+1)}\,b^{2(2B+1)}.$$

Clearly, J must be odd. So, let $J=2C+1$ where $C\geq 0$. Then, from the dimensional considerations, we have

$$I=2B+2^{\alpha-1}(2A+1)-3C-1.$$

Also, the coefficient of $a^{2^{\alpha}(2A+1)}\,b^{2(2B+1)}$ in $(a+b)\,W_2^{I}\,W_3^{J}$ is given by

$$G=\binom{2B+1+(2^{\alpha-1}(2A+1)-C)}{2B+2-(2^{\alpha-1}(2A+1)-C)}.$$

Thus, it is enough to find a value of C for which $I>0, J>0$ and $G = 1$. Let $\left[\frac{2(2B+1)}{2^{\phi}}\right]=1$; [ ] being the greatest integer function. Clearly, under the given conditions $\phi \geq \theta+3$.

Case 1 — $\left[\frac{2^{\alpha}(2A+1)}{2^{\phi}}\right]=1$, and $\left[\frac{2^{\alpha}(2A+1)}{2^{\theta+2}}\right]\cdot 2^{\theta+2}=2^{\phi}$.

Put $C=2^{\alpha-1}(2A+1)+2B+2-2^{\phi}$. Clearly, this value of C satisfies our requirements once we note that

$$2^{\phi}<4B+4\leq 2^{\phi-1}-2^{\theta+2}$$

and

$$2^{\phi}\leq 2^{\alpha}(2A+1)\leq 2^{\phi}+2^{\theta+2}-4.$$

Case 2 — $\left[\frac{2^{\alpha}(2A+1)}{2^{\phi}}\right]=1$, but $\left[\frac{2^{\alpha}(2A+1)}{2^{\theta+2}}\right]\cdot 2^{\theta+2}\neq 2^{\phi}$.

In this case, we take $C=2^{\alpha-1}(2A+1)-2^{\theta+1}-2^{\phi-1}$. Then once again our requirement is fulfilled, noting that

$$2^{\theta+2}=2^{\phi}\leq 2^{\alpha}(2A+1)<2^{\phi+1}$$

and

$$2B+1=2^{\theta+1}-1+2^{\phi-1}+P,$$

where P is a sum of powers of 2 between $2^{\theta+2}$ and $2^{\phi-2}$.

Case 3 — $\left[\frac{2^{\alpha}(2A+1)}{2^{\phi-1}}\right]\leq 1$.

We first consider the following subcases :

Subcase 3.1 — $\left[\frac{2^{\alpha}(2A+1)}{2^{\phi-1}}\right]=0$.

Subcase 3.2 — $\left[\frac{2^{\alpha}(2A+1)}{2^{\phi-1}}\right]=1$, and either $\left[\frac{2^{\alpha}(2A+1)}{2^{\theta+2}}\right]\cdot 2^{\theta+2}=2^{\phi-1}$

or $$\left[\frac{2(2B+1)}{2^{\theta+2}}\right]\cdot 2^{\theta+2}=2^{\phi+1}-2^{\theta+3}.$$

In these subcases it is easy to see that $C=2^{\alpha-1}(2A+1)-2^{\theta+1}$ serves the purpose. One simply notes that $2(2B+1)-2^{\alpha}(2A+1)>(2B+1)-3.2^{\theta+1}+1$ and $2B+1=2^{\theta+1}\cdot odd-1$.

Finally, we consider the following subcase

Subcase 3.3 — $\left[\frac{2^{\alpha}(2A+1)}{2^{\phi-1}}\right]=1, \left[\frac{2^{\alpha}(2A+1)}{2^{\theta+2}}\right]\cdot 2^{\theta+2}\neq 2^{\phi-1}$,

and
$$\left[\frac{2(2B+1)}{2^{\theta+2}}\right]\cdot 2^{\theta+2}\neq 2^{\phi+1}-2^{\theta+3}.$$

In this case, we take $C=2^{\alpha-1}(2A+1)-2^{\theta+1}-2^{\lambda-1}$, where λ is given by
$$\left[\frac{2(2B+1)}{2^{\lambda}}\right]\cdot 2^{\lambda}=2^{\phi+1}-2^{\lambda+1}.$$

Clearly, $\theta+2<\lambda<\phi$. That, this value of C satisfies our requirements, follows immediately once we observe that
$$2^{\theta+2}+2^{\phi-1}\leq 2^{\alpha}(2A+1)<2^{\phi},$$
and
$$2B+1=(2^{\theta+1}-1)+(2^{\phi}-2^{\lambda})+R,$$
where R is a sum of certain powers of 2 between $2^{\theta+2}$ and $2^{\lambda-2}$.

Thus the theorem is completely proved. □

We shall now show that the above theorem holds for $\alpha=1$ also, and this will complete our quest for the Milnor manifolds that bound.

Theorem 3.4 — *Let* $B+1=2^{\theta}\cdot$ *odd. Then* $H(2(2A+1), 2(2B+1))$ *does not bound if* $2(2A+1)\geq 2^{\theta+2}$.

PROOF : We first show that
$$W_{2^{\theta+3}-2}=a^{2^{\theta+2}-1}b^{2^{\theta+2}-1}+\text{ terms divisible by } a^{2^{\theta+2}}. \qquad \dots (1)$$

We have
$$W_{2^{\theta+3}-2}=\Sigma\binom{4A+3}{f}\binom{4B+3}{g}\binom{i+j}{i}a^{t}b^{2^{\theta+3}-2-t}, \qquad \dots (2)$$
where the sum is over all f, g, i, j, t such that $f+i=t$, $g+j=2^{\theta+3}-2-t$, and $0\leq t\leq 2^{\theta+3}-2$. Note that

(*i*) $\binom{4B+3}{g}=1$ if and only if $g<2^{\theta+2}$,

and

(*ii*) $\binom{i+j}{i}=1$ if and only if i and j have no power of 2 common in their 2-*addic* expansion.

Let us call those values of f, g, i, j as acceptable values for which the corresponding mod 2 binomial coefficients in the *R.H.S.* of (2) are nonzero. Now, let $t=2^{\theta+2}-1$. Then by (*i*), the coefficient of $a^{2^{\theta+2}-1}\,b^{2^{\theta+2}-1}$ in the *R.H.S.* of (2) is given by

$$L_{2^{\theta+2}-1} = \Sigma \begin{pmatrix} 4A+3 \\ f \end{pmatrix}\begin{pmatrix} i+j \\ i \end{pmatrix}.$$

Clearly, $f = 0$ is acceptable. In that case $i = 2^{\theta+2} - 1$, and conseqently, by (*ii*), $j = 0$ is the only acceptable value. On the other hand if $f = 2^{\alpha_1} + \ldots + 2^{\alpha_s}$, where $s \geq 1$ and $0 \leq \alpha_1 < \ldots < \alpha_s \leq \theta + 1$, is an acceptable value then $i = (2^{\theta+2} - 1) - (2^{\alpha_1} + \ldots + 2^{\alpha_s})$, and so, by (*ii*), the acceptable values of j are the elements of $S(2^{\alpha_1}, \ldots, 2^{\alpha_s})$, the set containing zero and the sums of all possible combinations of $2^{\alpha_1}, \ldots, 2^{\alpha_s}$. Note that $\# \, S(2^{\alpha_1}, \ldots, 2^{\alpha_s}) = 2^s$. Hence, it follows that $L_{2^{\theta+2}-1} = 1$.

Now, consider any t such that $0 \leq t < 2^{\theta+2} - 1$. Then, $2^{\theta+3} - 2 - t \geq 2^{\theta+2}$. Let L_t denote the coefficient of $a^t\, b^{2^{\theta+3}-2-t}$ in the *R.H.S.* of (2). In this case, for any acceptable value of f, we have $i = (2^{\theta+2} - 1) - (2^{\alpha_1} + \ldots + 2^{\alpha_s})$, where $s \geq 1$ and $0 \leq \alpha_1 < \ldots < \alpha_s \leq \theta + 1$. If $2^{\theta+3} - 2 - t = 2^{\theta+2}$, then, in view of (*i*) and (*ii*), the acceptable values of j are $2^{\theta+2}$ and the nonzero elements of $S(2^{\alpha_1}, \ldots, 2^{\alpha_s})$, and so we have $L_t = 0$. Let $2^{\theta+3} - 2 - t = 2^{\theta+2} + 2^{\beta_1} + \ldots + 2^{\beta_r}$, where $r \geq 1$ and $0 \leq \beta_1 < \ldots < \beta_r \leq \theta + 1$. Then, by (*i*) and (*ii*), the acceptable values of j are the elements of $S \cup T$ where S is the set

$$\{\sigma \mid \sigma \in S(2^{\alpha_1}, \ldots, 2^{\alpha_s}), \text{ and } \sigma \geq 1 + 2^{\beta_1} + \ldots + 2^{\beta_r}\}$$

and T is the set

$$\{2^{\theta+2} + \rho \mid \rho \in S(2^{\alpha_1}, \ldots, 2s^{\alpha_s}), \text{ and } \rho \leq 2^{\beta_1} + \ldots + 2^{\beta_r}\}.$$

Clearly, $(S \cup T) = S(2^{\alpha_1}, \ldots, 2^{\alpha_s}) = 2^s$. **Hence, we again have $L_t = 0$. This establishes the equality in (1).**

Next, we note that $W_2 = a^2 + ab + b^2$, $W_3 = ab\,(a+b)$, and the total Stiefel-Whitney class

$$W(H(2(2A+1), 2(2B+1)))$$

$$= (1 + W_2 + W_3)^{-1}\,(1+a)^{2^{\phi+2}\cdot odd}\,(1+b)^{2^{\theta+2}\cdot odd}, \qquad \ldots (1)$$

where ϕ is given by $A + 1 = 2^{\phi} \cdot odd$.

Case 1 — $\phi < \theta$.

In this case, by (3), we have

$$W_{2^{\phi+2}} + \Sigma\, W_2^i\, W_3^j = a^{2^{\phi+2}}, \qquad \ldots (4)$$

where the sum is over all $i, j >= 0$ such that $2i + 3j = 2^{\phi+2}$ and $\begin{pmatrix} i+j \\ i \end{pmatrix} = 1$. Then it is easy to see that

$$(a+b)\, V_{2^{\phi+2}}\, W_3^{2^{\phi+2}-1}\, W_{2^{\theta+3}-2}\, W_2^R = a^{4A+2}\, b^{4B+2},$$

where $V_{2^{\phi+2}}$ is the L.H.S. of (4),

$$I = \frac{4A+4-2^{\theta+2}-2^{\phi+2}}{2^{\phi+2}},$$

and

$$R = 2A+2b+4-2^{\phi+1}\cdot(I+3)-2^{\theta+2}.$$

Note here that $4A+2 \geq 2^{\theta+2}$, $4A+4 = 2^{\phi+2}\cdot odd$ and $A < B$.

Case 2 — $\phi \geq \theta$.

In this case, by (3), we have

$$W_{2^{\theta+2}} + \Sigma\, W_2^i\, W_3^j = \begin{cases} a^{2^{\theta+2}}\, b^{2^{\theta+1}}, & \text{if } \phi = \theta \\ a^{2^{\theta+1}}\, (a^{2^{\theta+1}} + b^{2^{\theta+1}}, & \text{if } \phi > \theta \end{cases} \qquad \ldots (5)$$

where the sum is over all i, $j > 0$ such that $2i+3j = 2^{\theta+2}$ and $\binom{i+j}{i} = 1$. Then, once again it is easy to see that

$$(a+b)\, V_{2^{\theta+2}}^J\, W_3^{\theta+2-1}\, W_{2^{\theta+3}-2}\, W_2^R = a^{4A+2}\, b^{4B+2},$$

where $V_{2^{\theta+2}}$ is the L.H.S. of (5),

$$I = \frac{4A+4-2^{\theta+3}}{2^{\theta+1}},$$

and

$$R = 2A+2B+4-2^{\theta+1}\cdot I-2^{\theta+3}-2^{\theta+1}.$$

Note here that $4A+2 \geq 2^{\theta+2}$, $4A+4$ is a positive multiple of $2^{\theta+2}$, and $B - A$ is a positive multiple of 2^{θ}.

Thus, in both the cases we see that $H(2(2A+1), 2(B+1))$ does not bound. This completes the proof. □

It is now routine matter to see that the combination of all the results proved so far yields Theorem 1.1.

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