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ON SOME NEW SPACES OF LACUNARY STRONGLY σ-CONVERGENT SEQUENCES DEFINED BY ORLICZ FUNCTIONS

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The main object of this paper is to introduce and study a new concept of lacunary strong σ-convergence with respect to an Orlicz function M. We examine some topological properties of the resulting sequence spaces and establish some elementary connections between lacunary strong σ-convergence and lacunary strong σ-convergence with respect to an Orlicz function which satisfies Δ_2-condition. We also give the relation between strong σ-convergence with respect to an Orlicz function and lacunary strong σ-convergence with respect to an Orlicz function.

Key Words : Lacunary Sequence; Orlicz Function; Invariant Mean; Strong Almost Convergence; Sequence Space; Δ_2-condition.

1. INTRODUCTION

Let l_∞ and c denote the Banach space of real bounded and convergent sequences $x = (x_k)$ normed by $\|x\| = \sup_k |x_k|$, respectively. A sequence $x = (x_k) \in l_\infty$ is said to be almost convergent if all of its Banach limits[1] coincide. Let $\hat{c}$ denote the space of all almost convergent sequences. Lorentz[9] proved that

$$\hat{c} = \{x \in l_\infty : \lim_{m \to \infty} t_{mn}(x) \text{ exists, uniformly in } n\},$$

where

$$t_{mn}(x) = \frac{x_n + x_{n+1} + \ldots.. + x_{n+m}}{m+1}.$$

The space $[\hat{c}]$ of strongly almost convergent sequences was introduced by Maddox[12] and also independently by Freedman *et al.*[4] as follows :

$$[\hat{c}] = \{x \in l_\infty : \lim_{m \to \infty} t_{mn}(|x - le|) = 0, \text{ uniformly in } n, \text{ for some } l\},$$

where $e = (1, 1, 1, \ldots)$.

Schaefer[20] defines the σ-convergence as follows :

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on l_∞ is said to be an invariant mean or a σ-mean if and only if,

(i) $\phi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,

(ii) $\phi(e) = 1$ and

(iii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in l_\infty$.

Let V_σ denote the set of bounded sequences which have unique σ-mean. If $x \in V_\sigma$ and $\phi(x) = l$, then we write $l = V_\sigma - \lim x$. In case σ is the translation mapping $n \to n+1$, a σ-mean reduces to the unique Banach limit and V_σ reduces to $\hat{c}$. We denote by V_σ the space of σ-convergent sequences. It is known[20] that $x \in V_\sigma$ if and only if,

$$\frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} \to \text{a limit}$$

as $m \to \infty$, uniformly in n. Here $\sigma^k(n)$ denotes the kth iterate of the mapping σ at n.

A σ-mean extends the limit functional on c in the sense that $\phi(x) = \lim x$ for all $x \in c$ if and only if σ has no finite orbits, that is to say, if and only if, for all $n > 0$, $k \geq 1$ (see [14]) $\sigma^k(n) \neq n$.

Just as the concept of almost convergence led naturally to the concept of strong almost convergence, σ-convergence leads naturally to the concept of strong σ-convergence. A sequence $x = (x_k)$ is said to be strongly σ-convergent (see [15]) if there exists a number l such that

$$\frac{1}{m} \sum_{k=1}^{m} |x_{\sigma^k(n)} - l| \to 0 \qquad \ldots (*)$$

as $m \to \infty$, uniformly in n. We write $[V_\sigma]$ as the set of all strongly σ-convergent sequences. When $(*)$ holds, we write $[V_\sigma] - \lim x = l$. Taking $\sigma(n) = n+1$, we obtain $[V_\sigma] = [\hat{c}]$ so that strong σ-convergence generalizes the concept of strong almost convergence. Note that $c \subset [V_\sigma] \subset V_\sigma \subset l_\infty$.

By a lacunary sequence $\theta = (k_r)$; $r = 0, 1, 2, \ldots$, where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and we let $h_r = k_r - k_{r-1}$. The ratio k_r/k_{r-1} will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman *et al.*[4] as follows :

$$N_\theta = \left\{ x = (x_k) : \lim_{r \to \infty} h_r^{-1} \sum_{k \in I_r} |x_k - l| = 0 \text{ for some } l \right\}.$$

There is a strong connection[4] between N_θ and the space ω of strongly Cesaro summable sequences, which is defined by

$$\omega = \left\{ x = (x_k) : \lim_{n \to \infty} n^{-1} \sum_{k=1}^{n} |x_k - l| = 0 \text{ for some } l \right\}.$$

In the special case where $\theta = (2^r)$, we have $N_\theta = \omega$.

Recently, Das and Mishra[2] have introduced the space AC_θ of lacunary almost convergent sequences and the space $|AC_\theta|$ of lacunary strongly almost convergent sequences, as follows :

$$AC_\theta = \left\{ x = (x_k) : \lim_{r \to \infty} h_r^{-1} \sum_{k \in I_r} (x_{k+n} - l) = 0 \text{ for some } l, \text{ uniformly, in } n \right\}$$

and

$$|AC_\theta| = \left\{ x = (x_k) : \lim_{r \to \infty} h_r^{-1} \sum_{k \in I_r} |x_{k+n} - l| = 0 \text{ for some } l, \text{ uniformly, in } n \right\}.$$

Note that in the special case where $\theta = (2^r)$, we have $AC_\theta = c$ and $|AC_\theta| = [c]$.

Quite recently, the concept of lacunary strong σ-convergence was introduced by Savas[19] which is a generalization of the idea of lacunary strong almost convergence due to Das and Mishra[2]. If $[V_\sigma^\theta]$ denotes the set of all lacunary strongly σ-convergent sequences, then Savas[19] defined

$$\left[V_\sigma^\theta \right] = \left\{ x = (x_k) : \lim_{r \to \infty} h_r^{-1} \sum_{k \in I_r} |x_{\sigma^k(n)} - l| = 0 \text{ for some } l, \text{ uniformly in } \right\} .$$

Note that for $\sigma(n) = n + 1$, the space $\left[v_\sigma^\theta \right]$ is the same as $|AC_\theta|$. We write $\left[V_\sigma^\theta \right] = \left[v_\sigma^\theta \right]_0$ in the case $l = 0$.

Recall[5,8] that an Orlicz function M is a continuous, convex, non-decreasing function defined for $x \geq 0$ such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$.

Lindenstrauss and Tzafriri[8] used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{ x = (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho} \right) < \infty \text{ for some } \rho > 0 \right\}.$$

The space l_M with the norm

$$\| x \| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho} \right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. Lindenstrauss and Tzafriri proved that every Orlicz sequence space l_M contains a subspace isomorphic to l_p for some $p \geq 1$, thereby answering a general conjecture that every infinite dimensional Banach space contains a closed subspace isomorphic to c_0 or some l_p, positively for a class of spaces (see [7] and [13] for discussion of this and related conjectures). For $M(x)=x^p$; $1 \le p < \infty$, the spaces l_M coincide with the classical sequence spaces l_p.

Recently, Parashar and Choudhary[17] have introduced and examined some properties of four sequence spaces defined by using an Orlicz function M, which generalized the well-known Orlicz sequence space l_M and strongly summable sequence spaces $[C, 1, p]$, $[C, 1, p]_0$ and $[C, 1, p]_\infty$. It may be noted here that the spaces of strongly summable sequences were discussed by Maddox[10] Nuray and Gülcü[16], Demirci[3] and others have also used an Orlicz function to construct some sequence spaces.

The main object of this paper is to introduce and study a new concept of lacunary strong σ-convergence with respect to an Orlicz function M. We examine some topological properties of the resulting $\left[V_\sigma^\theta, M, p\right]$ spaces (see Definition 1.1 below) and establish some elementary connections between lacunary strong σ-convergence and lacunary strong σ-convergence with respect to an Orlicz function which satisfies ∇_2 condition..

We also give the relation between strong σ-convergence with respect to an Orlicz function lacunary strong σ-convergence with respect to an Orlicz function.

We now introduce the generalizations of the spaces of lacunary strongly σ-convergent sequences.

Definition 1.1 — Let M be an Orlicz function and $p=(p_k)$ be any sequence of strictly positive real numbers. We define the following sequence spaces

$$\left[V_\sigma^\theta, M, p\right] = \left\{ x=(x_k) : \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} \left[M\left(\frac{|x_{\sigma^k(n)} - l|}{\rho} \right) \right]^{p_k} = 0 \right.$$

uniformly in n, for some l and $\rho > 0\}$

$$\left[V_\sigma^\theta, M, p\right]_0 = \left\{ x=(x_k) : \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} = 0 \right.$$

uniformly in n, for some $\rho > 0\}$

and

$$\left[V_\sigma^\theta, M, p\right]_\infty = \left\{ x=(x_k) : \sup_{r,n} h_r^{-1} \sum_{k\in I_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} < \infty \text{ for some } \rho > 0 \right\}.$$

We denote $\left[V_\sigma^\theta, M, p\right]$, $\left[V_\sigma^\theta, M, p\right]_0$ and $\left[V_\sigma^\theta, M, p\right]_\infty$ as $\left[V_\sigma^\theta, M\right]$, $\left[V_\sigma^\theta, M\right]_0$ and $\left[V_\sigma^\theta, M\right]_\infty$ when $p_k = 1$ for all k. If $x \in \left[V_\sigma^\theta, M\right]$ we say that x is lacunary strongly σ-convergent with respect to the Orlicz function M.

Some well-known spaces are obtained by specializing σ, θ, M and p.

(i) If $M(x) = x$, $\sigma(n) = n+1$, $\theta = (2^r)$ and $p_k = 1$ for all k, then $\left[V_\sigma^\theta, M, p\right] = [\hat{c}]$, $\left[V_\sigma^\theta, M, p\right]_0 = [\hat{c}_0]$ (Freedman *et al.*[4], Maddox[12])

(ii) If $M(x) = x$, $\theta = (2^r)$ and $p_k = 1$ for all k, then $\left[V_\sigma^\theta, M, p\right] = [V_\sigma]$ (Mursaleen[14])

(iii) If $M(x) = x$, $\sigma(n) = n+1$ and $p_k = 1$ for all k, then $\left[V_\sigma^\theta, M, p\right] = |AC_\theta|$ (Das and Mishra[2])

(iv) If $\theta = (2^r)$ then $\left[V_\sigma^\theta, M, p\right] = [V_\sigma, M]_{(p_k)}$, $\left[V_\sigma^\theta, M, p\right]_0 = [V_\sigma, M]^0_{(p_k)}$, $\left[V_\sigma^\theta, M, p\right]_\infty = [V_\sigma, M]^\infty_{(p_k)}$ (Nuray and Gülcü[16])

(v) If $M(x) = x$ and $\theta = (2^r)$ then $\left[V_\sigma^\theta, M, p\right] = [V_\sigma]_{(p_k)}$, $\left[V_\sigma^\theta, M, p\right]_0 = [V_\sigma]^\theta_{(p_k)}$, $\left[V_\sigma^\theta M, p\right]_\infty = [V_\sigma]^\infty_{(p_k)}$ (Savas[18])

(vi) If $M(x) = x$ and $p_k = 1$ for all k, then $\left[V_\sigma^\theta, M, p\right] = \left[V_\sigma^\theta\right]$ (Savas[19])

2. Main Results

In this section we examine some topological properties of $\left[V_\sigma^\theta, M, p\right]$ spaces and investigate some inclusion relations between these spaces.

Theorem 2.1 — *For any Orlicz function M and a bounded sequence $p = (p_k)$ of strictly positive real numbers, $\left[V_\sigma^\theta, M, p\right]$, $\left[V_\sigma^\theta, M, p\right]_0$ and $\left[V_\sigma^\theta, M, p\right]_\infty$ are linear spaces over the set of complex numbers.*

PROOF : We shall prove only for $\left[V_\sigma^\theta, M, p\right]_0$. The others can be treated similarly. Let $x, y \in \left[V_\sigma^\theta, M, p\right]_0$ and $\alpha, \beta \in C$. In order to prove the result we need to find some $\rho_3 > 0$ such that $\lim_r h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|\alpha x_{\sigma^k(n)} + \beta y_{\sigma^k(n)}|}{\rho_3}\right)\right]^{p_k} = 0$, uniformly in n. Since $x, y \in \left[V_\sigma^\theta, M, p\right]_0$, there exist positive ρ_1 and ρ_2 such that

$$\lim_r h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho_1}\right)\right]^{p_k} = 0$$

and

$$\lim_r h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|y_{\sigma^k(n)}|}{\rho_2} \right) \right]^{p_k} = 0, \text{ uniformly in } n.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is non-decreasing and convex,

$$h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|\alpha x_{\sigma^k(n)} + \beta y_{\sigma^k(n)}|}{\rho_3} \right) \right]^{p_k} \le h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|\alpha x_{\sigma^k(n)}|}{\rho_3} + \frac{|\beta y_{\sigma^k(n)}|}{\rho_3} \right) \right]^{p_k}$$

$$\le h_r^{-1} \sum_{k \in I_r} \frac{1}{2^{p_k}} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho_1} \right) + M\left(\frac{|y_{\sigma^k(n)}|}{\rho_2} \right) \right]^{p_k}$$

$$< h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho_1} \right) + M\left(\frac{|y_{\sigma^k(n)}|}{\rho_2} \right) \right]^{p_k}$$

$$\le C h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho_1} \right) \right]^{p_k} + C h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|y_{\sigma^k(n)}|}{\rho_2} \right) \right]^{p_k}$$

$\to 0$ as $r \to \infty$, uniformly in n, where C = ma. $(1, 2^{H-1})$, $H =$ sup p_k; so that $\alpha x + \beta y \in \left[V_\sigma^\theta, M, p \right]_0$. This proves that $\left[V_\sigma^\theta, M, p \right]_0$ is linear.

Theorem 2.2 — *For any Orlicz function M and a bounded sequence $p = (p_k)$ of strictly positive real numbers, $\left[V_\sigma^\theta, M, p \right]_0$ is a topological linear space, totally paranormed by*

$$g(x) = \inf \left\{ \rho^{p_r/H} : \left(h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} \right)^{1/H} \le 1, r = 1, 2, \ldots; n = 1, 2, \ldots \right\}$$

where H = max (1, sup$_k$ p_k).

PROOF : Clearly $g(x) = g(-x)$. By using Theorem 2.1 for $\alpha = \beta = 1$, we get $g(x + y) \le g(x) + g(y)$. Since $M(0) = 0$, we get $\inf \{\rho^{p_r/H}\} = 0$ for $x = 0$.

Conversely, suppose $g(x) = 0$, then

$$\inf \left\{ \rho^{p_r/H} : \left(h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} \right)^{1/H} \le 1 \right\} = 0.$$

This implies that for a given $\varepsilon > 0$, there exists some $\rho_\varepsilon (0 < \rho_\varepsilon < \varepsilon)$ such that

$$\left(h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho_\varepsilon} \right) \right]^{p_k} \right)^{1/H} \leq 1.$$

Thus

$$\left(h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\varepsilon} \right) \right]^{p_k} \right)^{1/H} \leq \left(h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho_\varepsilon} \right) \right]^{p_k} \right)^{1/H} \leq 1,$$

for each r and n.

Suppose that $x_{\sigma^i(j)} \neq 0$ for some i and j. Let $\varepsilon \to 0$, then $\left(\frac{|x_{\sigma^i(j)}|}{\varepsilon} \right) \to \infty$.

It follows that

$$\left(h_r^{-1} \sum_{i \in I_r} \left[M\left(\frac{|x_{\sigma^i(j)}|}{\varepsilon} \right) \right]^{p_i} \right)^{1/H} \to \infty \text{ which is a contradiction.}$$

Therefore $x_{\sigma^i(j)} = 0$ for each i and j. Finally, we prove that scalar multiplication is continuous. Let λ be any complex number. By definition,

$$g(\lambda x) = \inf \left\{ \rho^{p_r/H} : \left(h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|\lambda x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} \right]^{1/H} \leq 1, r = 1, 2, \ldots; n = 1, 2, \ldots$$

Then

$$g(\lambda x) = \inf \left\{ (|\lambda| s)^{p_r/H} : \left(h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{s} \right) \right]^{p_k} \right)^{1/H} \leq 1, r = 1, 2, \ldots; n = 1, 2, \ldots \right\}$$

where $s = \rho / |\lambda|$. Since $|\lambda|^{p_r} \leq \max(1, |\lambda|^{\sup p_r})$, we have

$$g(\lambda x) \leq (\max(1, |\lambda|^{\sup p_r}))^{1/H} \inf$$

$$\left\{ s^{p_r/H} : \left(h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{s} \right) \right]^{p_k} \right)^{1/H} \leq 1, r = 1, 2, \ldots; n = 1, 2, \ldots \right\}$$

which converges to zero as x converges to zero in $\left[V_\sigma^\theta, M, p \right]_0$.

Now suppose $\lambda_n \to 0$ and x is fixed in $\left[V_\sigma^\theta M, p \right]_0$. For arbitrary $\varepsilon > 0$, let N be a positive integer such that

$$h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} < (\varepsilon/2)^H \text{ for some } \rho > 0, r > N \text{ and all } n.$$

This implies that

$$\left(h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} \right)^{1/H} < \varepsilon/2 \text{ for some } \rho > 0, r > N \text{ and all } n.$$

Let $0 < |\lambda| < 1$, using convexity of M, for $r > N$ and all n, we get

$$h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|\lambda x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} < h_r^{-1} \sum_{k \in I_r} \left[|\lambda| M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} < (\varepsilon/2)^H.$$

Since M is continuous everywhere in $[0, \infty)$, then for $r \leq N$,

$$f(t) = h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|t x_{\sigma(x)^k}|}{\rho} \right) \right]^{p_k}$$

is continuous at 0. So there is $1 > \delta > 0$ such that $|f(t)| < (\varepsilon/2)^H$ for $0 < t < \delta$. Let K be such that $|\lambda_i| < \delta$ for $I > K$, then for $i > K, r \leq N$ and all n,

$$\left(h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|\lambda_i x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} \right)^{1/H} < \varepsilon/2.$$

Thus

$$\left(h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|\lambda_i x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} \right)^{1/H} < \varepsilon \text{ for } i > K \text{ and all } r \text{ and } n,$$

so that $g(\lambda x) \to 0\ (\lambda \to 0)$.

Definition 2.3[6] — An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists a constant $K > 0$ such that $M(2u) \leq K\, M(u), u \geq 0$.

It is easy to see that always $K > 2$. The Δ_2-condition is equivalent to the satisfaction of inequality $M(l\, u) \leq K(l)\, M(u)$ for all values of u and for $l > 1$.

Lemma 2.4 — Let M be an Orlicz function which satisfies Δ_2-condition and let $0 < \delta < 1$. Then for each $x \geq \delta$ we have $M(x) < Kx\, \delta^{-1} M(2)$ for some constant $K > 0$.

PROOF : Since M is non-decreasing and convex, and

$x < \delta^{-1} x < 1 + \delta^{-1} x$ for $x \geq \delta$, it follows that

$$M(x) < M(1 + \delta^{-1} x) = M\left(\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 2\, \delta^{-1} x\right) < \frac{1}{2} M(2) + \frac{1}{2} M(2\delta^{-1} x).$$

Since M satisfies Δ_2-condition, there is a constant $K > 2$ such that

$$M(2\delta^{-1} x) \leq \frac{1}{2} K\delta^{-1} x M(2),$$

therefore
$$M(x) < \frac{1}{2} K\delta^{-1} x\, M(2) + \frac{1}{2} K\, \delta^{-1} x M(2)$$

$$= K\delta^{-1} x\, M(2)$$ and hence the lemma.

Theorem 2.5 — *For any Orlicz function* M *which satisfies Δ_2-condition, we have*

$$\left[V_\sigma^\theta \right] \subseteq \left[V_\sigma^\theta, M \right].$$

PROOF : Let $x \in \left[V_\sigma^\theta \right]$ so that

$$A_r \equiv h_r^{-1} \sum_{k \in I_r} | x_{\sigma^k(n)} - l | \to 0 \text{ as } r \to \infty \text{ uniformly in } n, \text{ for some } l,$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \leq t \leq \delta$.
We can write

$$h_r^{-1} \sum_{k \in I_r} M(| x_{\sigma^k(n)} - l |) = h_r^{-1} \sum_{\substack{k \in I_r \\ | x_{\sigma^k(n)} - l | \leq \delta}} M(| x_{\sigma^k(n)} - l |) + h_r^{-1} \sum_{\substack{k \in I_r \\ | x_{\sigma^k(n)} - l | > \delta}} M(| x_{\sigma^{k}(n)} - l |)$$

$$< h_r^{-1} (h_r\, \varepsilon) + h_r^{-1} K\, \delta^{-1} M(2)\, h_r A_r,$$

by Lemma 2.4 Letting $r \to \infty$, it follows that $x \in \left[V_\sigma^\theta, M \right]$.

The method of the proof of Theorem 2.5 shows that, for any Orlicz function M which satisfies Δ_2-condition, we have $\left[V_\sigma^\theta \right]_0 \subseteq \left[V_\sigma^\theta, M \right]_0$ and $\left[V_\sigma^\theta \right]_\infty \subseteq \left[V_\sigma^\theta, M \right]_\infty$.

We now study the inclusions $[V_\sigma, M, p] \subseteq \left[V_\sigma^\theta, M, p \right]$ and $\left[V_\sigma^\theta, M, p \right] \subseteq [V_\sigma, M, p]$ under certain restrictions on $\theta = (k_r)$.

Theorem 2.6 — *Let $\theta = (k_r)$ be a lacunary sequence with lim $\inf_r q_r > 1$, then for any Orlicz function M, $[V_\sigma, M, p] \subseteq \left[V_\sigma^\theta, M, p \right]$, where*

$$[V_\sigma, M, p] = \left\{ x = (x_k) : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left[M\left(\frac{|x_{\sigma^k(n)} - l|}{\rho} \right) \right]^{p_k} = 0 \right.$$

uniformly in n, for some l, and $\rho > 0\}$

(We write $[V_\sigma, M, p] = [V_\sigma, M, p]_0$ in the case when $l = 0$).

PROOF : It is sufficient to show that $[V_\sigma, M, p]_0 \subseteq \left[V_\sigma^\theta, M, p \right]_0$; the general inclusion follows by linearity. Suppose lim $\inf_r q_r > 1$, then there exists $\delta > 0$ such that $q_r = (k_r/k_{r-1}) \geq 1 + \delta$ for all $r \geq 1$. Then for $x \in [V_\sigma, M, p]_0$, we write

$$A_r \equiv h_r^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k}$$

$$= h_r^{-1} \sum_{k=1}^{k_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} - h_r^{-1} \sum_{k=1}^{k_{r-1}} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k}$$

$$= \frac{k_r}{h_r} \left(k_r^{-1} \sum_{k=1}^{k_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} \right) - \frac{k_{r-1}}{h_r} \left(k_{r-1}^{-1} \sum_{k=1}^{k_{r-1}} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} \right)$$

Since $h_r = k_r - k_{r-1}$, we have

$$\frac{k_r}{h_r} \leq \frac{1+\delta}{\delta} \text{ and } \frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}.$$

The terms

$$k_r^{-1} \sum_{k-1}^{k_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} \text{ and } k_{r-1}^{-1} \sum_{k-1}^{k_{r-1}} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k}$$

both converge to zero uniformly in n, and it follows that A_r converges to 0 as $r \to \infty$, uniformly in n, that is, $x \in \left[V_\sigma^\theta, M, p \right]_0$

Theorem 2.7 — *Let* $\theta = (k_r)$ *be arlacunary sequence with lim* $\sup_r q_r < \infty$, *then for any Orlicz function* $M, \left[V_\sigma^\theta, M, p \right] \subseteq \left[V_\sigma, M, p \right]$.

PROOF : If $\lim \sup_r q_r < \infty$, there exists $B > 0$ such that $q_r < B$ for all $r \geq 1$. Let $x \in \left[V_\sigma^\theta, M, p \right]_0$ and $\varepsilon > 0$. There exists $R > 0$ such that for every $j \geq R$ and all n

$$A_j \equiv h_j^{-1} \sum_{k \in I_j} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} < \varepsilon.$$

We can also find $K > 0$ such that $A_j < K$ for all $j = 1, 2, \ldots$. Now let m be any integer with $k_{r-1} < m \leq k_r$, where $r > R$. Then

$$m^{-1} \sum_{k=1}^{m} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} \leq k_{r-1}^{-1} \sum_{k=1}^{k_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} = k_{r-1}^{-1}$$

$$\left\{ \sum_{k \in I_1} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} + \sum_{k \in I_2} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} + \ldots\ldots \sum_{k \in I_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} \right\}$$

$$= \frac{k_1}{k_{r-1}} k_1^{-1} \sum_{k \in I_1} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} + \frac{k_2 - k_1}{k_{r-1}} (k_2 - k_1)^{-1} \sum_{k \in I_2} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k}$$

$$+ \ldots + \frac{k_R - k_{R-1}}{k_{r-1}} (k_R - k_{R-1})^{-1} \sum_{k \in I_R} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k}$$

$$+ \ldots\ldots \frac{k_r - k_{r-1}}{k_{r-1}} (k_r - k_{r-1})^{-1} \sum_{k \in I_r} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k}$$

$$= \frac{k_1}{k_{r-1}} A_1 + \frac{k_2 - k_1}{k_{r-1}} A_2 + \ldots + \frac{k_R - k_{R-1}}{k_{r-1}} A_R + \frac{k_{R+1} - k_R}{k_{r-1}} A_{R+1} + \ldots\ldots \frac{k_r - k_{r-1}}{k_{r-1}} A_r$$

$$\leq \left(\sup_{j \geq 1} A_j \right) \frac{k_R}{k_{r-1}} + \left(\sup_{j \geq R} A_j \right) \frac{k_r - k_R}{k_{r-1}}$$

$$< K \frac{k_R}{k_{r-1}} + \varepsilon B.$$

Since $k_{r-1} \to \infty$ as $m \to \infty$, it follows that

$$m^{-1} \sum_{k=1}^{m} \left[M\left(\frac{|x_{\sigma^k(n)}|}{\rho} \right) \right]^{p_k} \to 0$$

uniformly in n and, consequently $x \in \left[V_{\sigma}^{\theta}, M, p \right]_0$.

The next result follows from Theorems 2.6 and 2.7.

Theorem 2.8 — *Let* $\theta = (k_r)$ *be a lacunary sequence with* $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$. *Then for any Orlicz function* M, $[V_{\sigma}, M, p] = \left[V_{\sigma}^{\theta}, M, p \right]$.

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