Indian J. pure appl. Math., 11(9) : 1125–1129, September 1980

HYPERSURFACES OF RECURRENT FINSLER SPACES

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(*Received* 12 *October* 1978; *after revision* 31 *December* 1979)

In this short note some properties of the hypersurfaces of recurrent Finsler spaces have been studied. Starting with Gauss equation corresponding to the relative curvature tensor of recurrent space and second fundamental tensor $\Omega^{*}_{\alpha\beta}$ of the hypersurface with respect to the secondary normal N^{*i} a number of conditions have been obtained so that hypersurface may be recurrent.

1. INTRODUCTION

Moór (1972) initiated the study of hypersurfaces of a recurrent Finsler space. Singh and Singh (1976) extended the investigation to umbilical subspaces of recurrent Finsler space. In this note we shall investigate some properties of the hypersurface of a Finsler space whose associate relative curvature tensor $\tilde{K}_{hijk}$ is recurrent. Our treatment is based on locally Minkowskian theory (Rund 1959, p. 189).

Consider an n-dimensional Finsler space F_n with fundamental metric function $F(x, \dot{x})$ which is positively homogeneous of first degree in $\dot{x}^{i}$(1) and satisfies usual conditions (Rund 1959). The metric tensor $g_{ij}(x, \dot{x})$ is given by

$$g_{ij}(x, \dot{x}) \overset{def}{=} \tfrac{1}{2}\,\dot{\partial}_i\dot{\partial}_j F^2(x, \dot{x}). \qquad ...(1.1)$$

A hypersurface F_{n-1} is defined by

$$x^i = x^i(u^\alpha) \qquad ...(1.2)$$

where u^α are Gaussian coordinates on F_{n-1}. The projection factor $B^i_\alpha \overset{def}{=} \partial x^i/\partial u^\alpha$ is such that the rank of the matrix B^i_α is $n-1$.

The metric tensor $g_{\alpha\beta}(u, \dot{u})$ of F_{n-1} is given by

$$g_{\alpha\beta}(u, \dot{u}) = g_{ij}(x, \dot{x})\, B^{ij}_{\alpha\beta} \qquad ...(1.3)$$

where $B^{i\cdots k}_{\alpha\cdots\varepsilon} = B^i_\alpha \ldots B^k_\varepsilon$.

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(1)The indices $i, j, \ldots$ take values $1, 2, \ldots, n$, $\alpha, \beta, \gamma, \ldots = 1, 2, \ldots, n-1$. Usual summation convention holds for repeated indices of the same type. Here we have used

$\dot{\partial}_i \equiv \partial/\partial\dot{x}^i$, $\equiv \dot{\partial}_\alpha/\partial\dot{u}^\alpha$, $\partial_i \equiv \partial/\partial x^i$, $\partial_\alpha \equiv \partial/\partial u^\alpha$, $x' = dx/ds$.

The vector $N^{*i}(x, x')$ satisfying the relations (Rund 1959)[2]

$$\left.\begin{array}{ll}\text{(a)} & g_{ij}B^i_\alpha N^{*j} = 0 \\ \text{(b)} & g_{ij}N^{*i}N^{*j} = \psi(x, x')\end{array}\right\} \qquad \ldots(1.4)$$

is called secondary normal to the hypersurface at the point. The inverse of B^i_α is B^α_i defined by

$$B^\alpha_i = g^{\alpha\beta}B^j_\beta g_{ij} \qquad \ldots(1.5)$$

so that

$$B^i_\alpha B^\beta_i = \delta^\beta_\alpha .$$

Rund (1959, p. 193) has defined $\overset{\circ}{\delta}$-operator. If X^i_α be a mixed tensor, then

$$\overset{\circ}{\delta}_\beta X^i_\alpha = \partial_\beta X^i_\alpha + \Gamma^i_{jk} X^j_\alpha B^k_\beta - \Gamma^\gamma_{\alpha\beta} X^i_\gamma{}^{(3)} \qquad \ldots(1.6)$$

In particular, we have

$$I^i_{\alpha\beta} = \overset{\circ}{\delta}_\beta B^i_\alpha = B^i_{\alpha\beta} + \Gamma^i_{jk} B^{jk}_{\alpha\beta} - \Gamma^\gamma_{\alpha\beta} B^i_\gamma \qquad \ldots(1.7)$$

where $B^i_{\alpha\beta} = \partial^2 x^i/\partial u^\alpha \partial u^\beta$.

For a tensor T^i_j and T^α_β of F_n and F_{n-1} respectively, we have

$$\left.\begin{array}{ll}\text{(a)} & \overset{\circ}{\delta}_\alpha T^i_j = T^i_{j;k} B^k_\alpha \\ \text{(b)} & \overset{\circ}{\delta}_\alpha T^\gamma_\beta = T^\gamma_{\beta;\alpha} .\end{array}\right\} \qquad \ldots(1.8)$$

Here (;) denotes the δ-covariant differentiation as given in (Rund 1959). Gauss's equation for the present case is given by (Rund 1959)

$$\widetilde{K}_{\alpha\lambda\gamma\beta}(u, u') = \psi(\Omega^*_{\alpha\gamma}\Omega^*_{\lambda\beta} - \Omega^*_{\alpha\beta}\Omega^*_{\lambda\gamma}) + \widetilde{K}_{hjlk}(x, x') B^{jhkl}_{\lambda\alpha\beta\gamma}$$
$$- N^{*l}C^*_{jlk} B^j_\lambda (B^k_\gamma \Omega^*_{\alpha\beta} - B^k_\beta \Omega^*_{\alpha\gamma}) \qquad \ldots(1.9)$$

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[2]The unit normal vector N^{*i} is not in general coincident with unit normal N^i of F_{n-1}. The relation between N^i and N^{*i} is given in (Rund 1959).

[3]For the sake of convenience, we have written Γ^i_{jk} for Cartan's connection Γ^{*i}_{jk} of F_n (Rund 1959, p. 193). Also $\Gamma^\alpha_{\beta\gamma}$ is corresponding induced connection parameter of F_{n-1} (Rund 1959, p. 160).

where we have (Rund 1959, pp. 194, 196)

$$I^{i}_{\alpha\beta}(x, x') = N^{*i}(x, x')\,\Omega^{*}_{\alpha\beta}(u, u') \qquad \text{...(1.10)}$$

$$C^{*}_{ij,\alpha}(x, x') = \overset{\circ}{\delta}_{\alpha} g_{ij}(x, x') \qquad \text{...(1.11)}$$

and

$$C^{*}_{ij,\alpha} \overset{def}{=} C^{*}_{ijk}\, B^{k}_{\alpha}\,.$$

Here $\widetilde{K}_{hijk}(x, x')$ is associate relative curvature tensor of F_n.

2. Recurrent F_n and Totally Geodesic F_{n-1}

A Finsler space F_n will be called recurrent (or more appropriately $\widetilde{K}$-recurrent) if there exists a non zero vector K_i such that

$$\widetilde{K}_{hjkl;m} = K_m \widetilde{K}_{hjlk}. \qquad \text{...(2.1)}$$

Taking covariant derivative of (1.9) with respect to u^{σ} and noting (1.7) and (1.8), we have

$$\begin{aligned}\widetilde{K}_{\alpha\lambda\gamma\beta;\sigma} = {} & \widetilde{K}_{hjlk;m} B^{jhklm}_{\lambda\alpha\beta\gamma\sigma} + \widetilde{K}_{hjlk}\,[B^{hkl}_{\alpha\beta\gamma}\, I^{j}_{\lambda\sigma} + B^{jkl}_{\lambda\beta\gamma}\, I^{h}_{\alpha\sigma} \\ & + B^{jhl}_{\lambda\alpha\gamma}\, I^{k}_{\beta\sigma} + B^{jhk}_{\lambda\alpha\beta}\, I^{l}_{\gamma\sigma}] + [\psi(\Omega^{*}_{\alpha\gamma}\,\Omega^{*}_{\lambda\beta} - \Omega^{*}_{\alpha\beta}\,\Omega^{*}_{\lambda\gamma}) \\ & - N^{*l} C^{*}_{jlk}\, B^{j}_{\lambda}\,(\Omega^{*}_{\alpha\beta}\, B^{k}_{\gamma} - \Omega^{*}_{\alpha\gamma}\, B^{k}_{\beta})]_{;\sigma}.\end{aligned} \qquad \text{...(2.2)}$$

Put

$$M_{\alpha\lambda\gamma\beta} = \psi(\Omega^{*}_{\alpha\gamma}\,\Omega^{*}_{\lambda\beta} - \Omega^{*}_{\alpha\beta}\,\Omega^{*}_{\lambda\gamma}) - N^{*l} C^{*}_{jlk}\, B^{j}_{\lambda}\,(\Omega^{*}_{\alpha\beta}\, B^{k}_{\gamma} - \Omega^{*}_{\alpha\gamma}\, B^{k}_{\beta}) \qquad \text{...(2.3)}$$

and

$$L_{\alpha\lambda\gamma\beta} = \widetilde{K}_{\alpha\lambda\gamma\beta} - M_{\alpha\lambda\gamma\beta}. \qquad \text{...(2.4)}$$

Then using (2.1) and consequently eliminating $\widetilde{K}_{hjlk} B^{jhkl}_{\lambda\alpha\beta\gamma}$ from (2.2) and (1.9) we have

$$\begin{aligned}L_{\alpha\lambda\gamma\beta;\sigma} = {} & K_{\sigma} L_{\alpha\lambda\gamma\beta} + \widetilde{K}_{hjlk}\,[B^{hkl}_{\alpha\beta\gamma}\, I^{j}_{\lambda\sigma} + B^{jkl}_{\lambda\beta\gamma}\, I^{h}_{\alpha\sigma} \\ & + B^{jhl}_{\lambda\alpha\gamma}\, I^{k}_{\beta\sigma} + B^{jhk}_{\lambda\alpha\beta}\, I^{l}_{\gamma\sigma}]\end{aligned} \qquad \text{...(2.5)}$$

where we have put

$$K_\sigma = K_m B^m_\sigma. \qquad ...(2.6)$$

It is easy to see that if $K_\sigma = 0$, the recurrence vector K_m is normal to the hypersurface F_{n-1}. Further, since B^m_σ may be regarded as $n-1$ vectors in F_{n-1} tangential to F_{n-1}, K_σ is tangential to F_{n-1}, being the linear combinations of these tangential vectors.

When the hypersurface is totally geodesic,

$$\Omega^*_{\alpha\beta} = 0. \qquad ...(2.7)$$

Consequently, we have in such a case

$$\text{(a)} \quad I^i_{\alpha\beta} = 0, \quad \text{(b)} \quad M_{\alpha\lambda\gamma\beta} = 0, \quad \text{(c)} \quad L_{\alpha\lambda\gamma\beta} = \tilde{K}_{\alpha\lambda\gamma\beta}. \qquad ...(2.8)$$

Thus we have the following theorems.

Theorem 2.1 — A totally geodesic hypersurface F_{n-1} of a recurrent F_n is itself recurrent with recurrence vector K_σ given by (2.6).

A hypersurface F_{n-1} is called symmetric if its curvature tensor is covariant constant, i.e. $\tilde{K}_{\alpha\lambda\beta\delta;\sigma} = 0$.

Hence we have the following:

Theorem 2.2 — A necessary and sufficient condition for a totally geodesic hypersurface F_{n-1} of a recurrent F_n to be symmetric is that the recurrence vector of F_n be normal to F_{n-1}.

The proof of the above theorem follows from (2.2), (2.6), (2.7) and (2.8a).

3. UMBILICAL F_{n-1}

A hypersurface F_{n-1} is called umbilical if its lines of curvature are indeterminate.

The condition for this is

$$\Omega^*_{\alpha\beta} = \rho g_{\alpha\beta} \qquad ...(3.1)$$

where $\rho = \Omega^*_{\alpha\beta}\, g^{\alpha\beta}/(n-1) = \dfrac{M^*}{(n-1)}$

M^* is the mean curvature of the hypersurface.

Assuming F_{n-1} to be umbilical, eqns. (2.4) and (2.5) take the form

$$\bar{L}_{\alpha\lambda\gamma\beta} = \tilde{K}_{\alpha\lambda\gamma\beta} - \bar{M}_{\alpha\lambda\gamma\beta} \qquad ...(3.2)$$

and

$$\bar{L}_{\alpha\lambda\gamma\beta;\sigma} = K_\sigma \bar{L}_{\alpha\lambda\gamma\beta} + \rho \tilde{K}_{hjlk} [B^{hkl}_{\alpha\beta\gamma} N^{*j} g_{\lambda\sigma} + B^{jkl}_{\lambda\beta\gamma} N^{*h} g_{\alpha\sigma}$$

$$+ B^{jhl}_{\lambda\alpha\gamma} N^{*k} g_{\beta\sigma} + B^{jhk}_{\lambda\alpha\beta} N^{*l} g_{\gamma\sigma}] \qquad \ldots(3.3)$$

respectively, where $\bar{L}$ and $\bar{M}$ are obtained from (2.4) and (2.3) respectively after substituting for $\Omega^*_{\alpha\beta}$ from (3.1).

Thus we have the following:

Theorem 3.1 — In case of an umbilical hypersurface F_{n-1} of a recurrent Finsler space F_n whose recurrence vector is not normal to F_{n-1}, the tensor $\bar{L}_{\alpha\lambda\gamma\beta}$ is recurrent if and only if

$$\tilde{K}_{hjlk} [B^{hkl}_{\alpha\beta\gamma} N^{*j} g_{\lambda\sigma} + B^{jkl}_{\lambda\beta\gamma} N^{*h} g_{\alpha\sigma} + B^{jhl}_{\lambda\alpha\gamma} N^{*k} g_{\beta\sigma}$$

$$+ B^{jhk}_{\lambda\alpha\beta} N^{*l} g_{\gamma\sigma}] = 0. \qquad \ldots(3.4)$$

The proof of the above theorem follows immediately from (3.3) since $\rho \neq 0$. Further, we have the following:

Theorem 3.2 — In case of an umbilical hypersurface F_{n-1} of a recurrent Finsler space F_n the tensor $\bar{L}_{\alpha\lambda\gamma\beta}$ is covariant constant iff (3.4) and

$$K_\sigma = 0 \qquad \ldots(3.5)$$

hold.

It may be remarked that (3.5) states that K_m is normal to the hypersurface.

ACKNOWLEDGEMENT

The authors are grateful to the referee for his valuable suggestions. One of them (H. S.) wishes to express his gratitude to U.G.C. for financial help under Faculty Improvement Programme.

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