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STABILITY OF FIXED POINT SETS AND COMMON FIXED POINTS OF FAMILIES OF MAPPINGS

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In the present work stability of fixed point sets of families of various types of mappings have been considered. In particular two types of convergence of mappings, pointwise convergence and uniform convergence are considered. The authors consider quasi-contractions introduced by Ciric, generalized Kannan-Reich type of mappings discussed by wong and mappings introduced by Husain and Sehgal. They also consider common fixed points of sequences of both single-valued and set-valued mappings. The works of Wong, Husain and Sehgal, Iseki and Bose and Mukherjee are extended.

Stability of fixed point sets of families of mappings of various types have been considered by Bonsall (1962), Nadler (1968, 1969), Markin (1973) and Ray (1973). Question considered by them is the following: Does the convergence of a sequence of mappings $\{f_i\}$ in a metric space to a mapping f imply the convergence of the sequence of their fixed points to the fixed point of f? Two types of convergence of mappings, pointwise convergence and uniform convergence, are considered. We have considered quasi-contractions discussed by Ciric (1974), generalized Kannan-Reich type of mappings discussed by Wong (1973) and mappings discussed by Husain and Sehgal (1975). Here we have obtained new results in this direction.

Also next we consider common fixed points of sequences of both single-valued and set-valued mappings. Recently Ray (1971), Wong (1973), Iseki (1974), Husain and Sehgal (1975) and Bose and Mukherjee (1977) proved some interesting theorems about common fixed points of mappings. Here we extend the works of Wong (1973) Husain and Sehgal (1975), Iseki (1974) and Bose and Mukherjee (1977).

Let X be a metric space with metric d.

§1. Ciric (1974) has considered self mappings of X of the following type.

Definition — A mapping $f : X \to X$ is said to be a quasi-contraction iff there exists a number q, $0 \leqslant q < 1$, such that

$$d(fx, fy) \leqslant q \max \{d(x, y); d(x, fx); d(y, fy); d(x, fy), d(y, fx)\}$$

holds for every $x, y \in X$.

The following theorem of Ciric (1974) will be used in the sequel.

Theorem 1.1 (Ciric 1974) — Let f be a quasi-contraction on a metric space X and let X be f-orbitally complete. Then

(a) f has a unique fixed point u in X;

(b) $\lim_{n\to\infty} f^n x = u$; and

(c) $d(f^n x, u) \leqslant \frac{q^n}{q-1} d(x, fx)$ for every $x \in X$.

We now prove the following theorems.

Theorem 1.2 — Let $\{f_n\}$ be a sequence of self mappings of X and each has at least one fixed point x_n and let $\{f_n\}$ converge to f uniformly where f is a quasi-contraction and X is f-orbitally complete. Then $\{x_n\}$ converges to x_0, the fixed point of f if $\sup d(x_n, x_0) < \infty$.

PROOF : We have $d(x_n, x_0) \leqslant d(f_n x_n, f x_n) + d(f x_n, f x_0)$, i.e.,

$$d(x_n, x_0) \leqslant d(f_n x_n, f x_n) + q \max \{d(f x_n, x_n); d(f x_0, x_0); d(f x_n, x_0); d(f x_0, x_n); d(x_n, x_0)\}.$$

Let $\epsilon = \limsup d(x_n, x_0)$. $\lim_{n\to\infty} d(f_n x_n, f x_n) = 0$ due to uniform convergence of $\{f_n\}$ to f. Then we have

$$\epsilon \leqslant q \max \{0; 0; \epsilon; \epsilon; \epsilon\} = q\epsilon < \epsilon \Rightarrow \epsilon = 0.$$

This proves the assertion.

Theorem 1.3 — Let $\{f_n\}$ be a sequence of quasi-contractions on a metric space X with same q and let $\{f_n\}$ converge pointwise to f, a quasi-contraction on X. Let X be f-orbitally and f_n-orbitally complete for $n = 1, 2, \ldots$. If x_n and x_0 are fixed points of f_n and f respectively, then $x_n \to x_0$.

PROOF : From Theorem 1.1 (c) we have

$$d(f_i^n x, x_i) \leqslant \frac{q^n}{1-q} d(x, f_i x) \;\forall\; x \in X. \quad i = 1, 2, \ldots .$$

Put $n = 0$ and $x = x_0$. Then

$$d(x_0, x_i) \leqslant \frac{1}{1-q} d(x_0, f_i x_0) = \frac{1}{1-q} d(f x_0, f_i x_0).$$

Taking limits as $i \to \infty$, we have $d(x_0, x_i) \to 0$.

§2. Wong (1973) considered generalized Kannan-Reich type of self-mappings of a complete matric space. We prove the following two theorems about such mappings.

Definition — A mapping $f : X \to X$ is said to be of generalized Kannan-Reich type if

$$d(fx, fy) \leqslant a_1 d(fx, x) + a_2 d(fy, y) + a_3 d(fx, y) + a_4 d(fy, x) + a_5\, d(x, y)$$

$\forall\, x, y \in X$ where a_i's are nonnegative numbers such that $\sum_{n=1}^{5} a_i < 1$. (Here $a_1 = a_2$ and $a_3 = a_4$ by symmetry).

Theorem 2.1 — Let $\{f_n\}$ be a sequence of self mappings of X having at least one fixed point x_n each and let $\{f_n\}$ converge uniformly to f, a mapping of generalized Kannan-Reich type. Let x_0 be the unique fixed point of f. Then $x_n \to x_0$.

PROOF : $d(x_n, x_0) \leqslant d(f_n x_n, fx_n) + d(fx_n, fx_0)$

$$\leqslant d(f_n x_n, fx_n) + a_1 d(fx_n, x_n) + a_2 d(fx_0, x_0)$$
$$+ a_3 d(x_0, fx_n) + a_4 d(x_n, fx_0) + a_5 d(x_0, x_n)$$
$$\leqslant (1 + a_1 + a_3)\, d(f_n x_n, fx_n) + (a_3 + a_4 + a_5)\, d(x_n, x_0)$$

i.e., $(1 - a_3 - a_4 - a_5)\, d(x_n, x_0) \leqslant (1 + a_1 + a_3)\, d(f_n x_n, fx_n)$.

Taking limit, we have $\lim_{n\to\infty} (1 - a_3 - a_4 - a_5)\, d(x_n, x_0) = 0$.

This implies that $\lim_{n\to\infty} d(x_n, x_0) = 0$ as $1 - a_3 - a_4 - a_5 > 0$.

Theorem 2.2 — Let $\{f_n\}$ be a sequence of mappings of generalized Kannan-Reich type and let $\{f_n\}$ converges pointwise to f, a generalized Kannan-Reich type mapping. Let x_n and x_0 be fixed points of f_n and f respectively. Then $x_n \to x_0$.

PROOF : This follows from Theorem 1.3 as such mappings are obviously quasi-contractions and X is a complete metric space.

Here we generalize a theorem of Iseki (1974).

Theorem 2.3 — Let $\{f_n\}$ be a sequence of self-mappings of X such that

$$d(f_i x, f_j y) \leqslant a_1 d(f_i x, x) + a_2 d(f_j y, y) + a_3 d(f_i x, y) + a_4 d(f_j y, x)$$
$$+ a_5 d(x, y) \quad (j > i)$$

for all $x, y \in X$ where a_1, a_2, a_3, a_4 and a_5 are nonnegative numbers and $\sum_{k=1}^{5} a_k < 1$ and $a_3 = a_4$. Then the sequence $\{f_n\}$ has a unique common fixed point.

PROOF : Let $x_0 \in X$. Let $x_n = f_n(x_{n-1})$, $n = 1, 2, \ldots$. Then we have

$$d(x_1, x_2) = d(f_1x_0, f_2x_1)$$
$$\leqslant a_1d(f_1x_0, x_0) + a_2d(f_2x_1, x_1) + a_3d(f_1x_0, x_1) + a_4d(f_2x_1, x_0) + a_5d(x_0, x_1)$$

i.e. $$d(x_1, x_2) \leqslant \frac{a_1 + a_4 + a_5}{1 - a_2 - a_4} d(x_0, x_1).$$

Also $$d(x_2, x_3) \leqslant \frac{a_1 + a_4 + a_5}{1 - a_2 - a_4} d(x_1, x_2) \leqslant r^2 d(x_0, x_1)$$

where $$r = \frac{a_1 + a_4 + a_5}{1 - a_2 - a_4} < 1.$$

Similarly $$d(x_n, x_{n+1}) \leqslant r^n d(x_0, x_1)$$

$$d(x_n, x_{n+p}) < \frac{r^n}{1 - r} d(x_0, x_1). \quad \text{For } p > 0$$

Thus $$d(x_n, x_{n+p}) \to 0 \text{ as } n \to \infty.$$

So $\{x_n\}$ is Cauchy in X. Since (X, d) is complete, there is an element x in X to which the sequence $\{x_n\}$ converges.

For any fixed n, we have

$$d(x, f_nx) \leqslant d(x, x_{m+1}) + d(x_{m+1}, f_nx) \quad (m > n)$$
$$= d(x, x_{m+1}) + d(f_nx, f_{m+1}x_m)$$
$$\leqslant d(x, x_{m+1}) + a_1d(f_nx, x) + a_2d(x_{m+1}, x_m) + a_3d(f_nx, x_m) + a_4d(x_{m+1}, x) + a_5d(x, x_m).$$

Taking limit as $m \to \infty$, we have

$$d(x, f_nx) \leqslant (a_1 + a_3)\, d(x, f_nx).$$

Therefore $d(x, f_nx) = 0$, i.e., x is a common fixed point of all f_n.

If $f_ny = y$ for some y in X and for every n, then

$$d(x, y) = d(f_nx, f_my),\ m > n$$
$$\leqslant (a_3 + a_4 + a_5)\, d(x, y).$$

Therefore $d(x, y) = 0$, i.e. $x = y$. So the common fixed point is unique.

§3. Husain and Sehgal (1975) have considered self mappings of a complete metric space X of the following type.

Definition — A self mapping of a complete metric space X is called Husian-Sehgal type if there exists a function $\phi : R^5 \to R^+$ which is continuous and non-decreasing in each co-ordinate and satisfies $\phi(t, t, a_1 t, a_2 t, t) < t$, $a_i \in \{1, 2\}$ with $a_1 + a_2 = 2$ such that for all $x, y \in X$

$$d(fx, fy) \leqslant \phi\,(d(x, fx), d(y, fy), d(y, fx), d(x, fy), d(x, y)).$$

Theorem 3.1 — Let $\{f_n\}$ be a sequence of mappings of Husain-Sehgal type with same ϕ and converge pointwise to f, a mapping of the same type. If x_n and x_0 are the fixed points of f_n and f respectively, then $x_n \to x_0$ if $\sup d(x_n, x_0) < \infty$.

PROOF : $d(x_n, x_0) \leqslant d(f_n x_n, f_n x_0) + d(f_n x_0, f x_0)$

$$\leqslant \phi\,[d(x_n, f_n x_n), d(f_n x_0, x_0), d(f_n x_n, x_0), d(f_n x_0, x_n), d(x_n, x_0)] + d(f_n x_0, f x_0).$$

Let $\epsilon = \limsup d(x_n, x_0)$.

Then, we have

$$\epsilon \leqslant \phi(0, 0, \epsilon, \epsilon, \epsilon) < \epsilon \Rightarrow \epsilon = 0.$$

Theorem 3.2 — Let $\{f_n\}$ and $\{g_n\}$ be two sequences of self mappings of X, a complete metric space, satisfying the following condition:

$$n \geqslant 1,\ d(f_n x, g_n y) \leqslant \phi\,(d(x, f_n x), d(y, g_n y), d(x, g_n y), d(y, f_n x), d(x, y))$$

where ϕ is a function as given in the definition of Husain-Sehgal type of mappings. Let $f_n \to f$ and $g_n \to g$ pointwise and let x_n be the common fixed point of f_n and g_n (which exists by a theorem of Husain and Sehgal (1975)). Then f and g have a common fixed point x_0 and $x_n \to x_0$ if $\sup d(x_n, x_0) < \infty$.

PROOF : It follows from Theorem 3.1.

Theorem 3.3 — Let $\{f_n\}$ and $\{g_n\}$ be two sequences of self mappings of X satisfying the condition of Theorem 3.2. Let $f_n \to f$ and $g_n \to g$ pointwise and let x_n be the common fixed point of f_n and g_n. If x_0 is a cluster point of $\{x_n\}$, then x_0 is the common fixed point of f and g.

PROOF : Let $x_{n_i} \to x_0$. Instead of using n_i we shall use i, For each $i \geqslant 1$

$$\begin{aligned} d(x_0, gx_0) &\leqslant d(x_0, x_i) + d(f_i x_i, g_i x_0) + d(g_i x_0, gx_0) \\ &\leqslant d(x_0, x_i) + \phi\,[d(f_i x_i, x_i), d(g_i x_0, x_0), \\ &\qquad d(f_i x_i, x_0), d(g_i x_0, x_0), d(x_0, x_i)] + d(g_i x_0, gx_0). \end{aligned}$$

Therefore as $i \to \infty$, we have

$$d(x_0, gx_0) \leqslant \phi(0, d(x_0, gx_0), 0, d(x_0, gx_0), 0) < d(x_0, gx_0)$$

as ϕ is continuous and non-decreasing in each coordinate. Then $d(x_0, gx_0) = 0$. Similarly $d(x_0, fx_0) = 0$.

§4. This section deals with set-valued mappings.

Let X be a bounded metric space with metric d and let H denote the Hausdorff metric on the space $CL(X)$, the space of non-empty closed subsets of X. We shall consider mappings from (X, d) into $(CL(X); H)$. Such mappings have been considered by Nadler (1969), Smithson (1971), Reich (1971), Assad and Kirk (1972) and Bose and Mukherjee (1977). We consider mappings which satisfy generalized Kannan-Reich type condition and also mappings which are contractive.

The following theorem is a generalization of Theorem 9 (Wong 1973).

Theorem 4.1 — Let $\{F_n\}$ be a sequence of multi-valued mappings of X into $CL(X)$ satisfying the following condition:

$$H(F_nx, F_ny) \leqslant a_n^1 D(F_nx, x) + a_n^2 D(F_ny, y) + a_n^3 D(F_nx, y)$$

$$+ a_n^4 D(F_ny, x) + a_n^5 D(x, y)$$

for all $x, y \in X$, where $a_n^j \geqslant 0$, $j = 1, 2, 3, 4$, and 5 and $\sum_{j=1}^{5} a_n^j < 1$.

Let $a_n^j \to a^j$ as $n \to \infty$ and $\sum_{j=1}^{5} a^j < 1$. Let $\{F_n\}$ converge to F_0 pointwise and let x_n be the fixed point of F_n. If x_0 is any cluster point of the sequence $\{x_n\}$, then $x_0 \in F_0x_0$.

PROOF: Let $x_{n_i} \to x_0$. For simplicity in notation, we write henceforward i in place of n_i. Then

$$D(x_0, F_0x_0) \leqslant d(x_0, x_i) + D(x_i, F_0x_0)$$

$$\leqslant d(x_0, x_i) + H(F_ix_i, F_0x_0) \text{ as } x_i \in F_ix_i,$$

$$\leqslant d(x_0, x_i) + H(F_ix_i, F_ix_0) + H(F_ix_0, F_0x_0).$$

So

$$D(x_0, F_0x_0) \leqslant d(x_0, x_i) + a_i^1 D(F_ix_i, x_i) + a_i^2 D(F_ix_0, x_0)$$

$$+ a_i^3 D(F_ix_i, x_0) + a_i^4 D(F_ix_0, x_i) + a_i^5 d(x_0, x_i)$$

$$+ H(F_ix_0, F_0x_0)$$

$$\leqslant d(x_0, x_i) + a_i^3 d(x_i, x_0) + a_i^4 d(x_i, x_0) + a_i^5 d(x_i, x_0)$$

$$+ a_i^2 H(F_ix_0, F_0x_0) + a_i^2 D(F_0x_0, x_0) + a_i^4 H(F_ix_0, F_0x_0)$$

$$+ a_i^4 D(F_0x_0, x_0) + H(F_ix_0, F_0x_0)$$

i.e., $$(1 - a_i^2 - a_i^4)\, D(x_0, F_0x_0) \leqslant (1 + a_i^3 + a_i^4 + a_i^5)\, d(x_0, x_i) + (1 + a_i^2 + a_i^4)\, H(F_ix_0, F_0x_0).$$

Taking limit as $i \to \infty$, we have

$$(1 - a^2 - a^4)\, D(x_0, F_0x_0) = 0$$

i.e., $$D(x_0, F_0x_0) = 0 \text{ as } (1 - a^2 - a^4) \neq 0.$$

Since F_0x_0 is closed, we have $x_0 \in F_0x_0$.

Remark: In this case $a_i^1 = a_i^2$ and $a_i^3 = a_i^4$ due to symmetry. Hence F_i has a fixed point by Corollary 2 to Theorem 1 (Bose and Mukherjee 1977).

Theorem 4.2 — Let $\{S_n\}$ and $\{T_n\}$ be two sequences of functions from X into $CL(X)$ which converge pointwise to the function S and T on X respectively. Suppose for each $n \geqslant 1$,

$$H(S_nx, T_ny) \leqslant a_n^1\, D(x, S_nx) + a_n^2\, D(T_ny, y) + a_n^3\, D(S_nx, y) + a_n^4\, D(T_ny, x) + a_n^5\, d(x, y) \text{ for all } x, y \in X$$

where $\sum_{j=1}^{5} a_n^j < 1$ and $a_n^1 = a_n^2$ or $a_n^3 = a_n^4$. Let $a_n^j \to a^j$ as $n \to \infty$ and $\sum_{i=1}^{5} a^i < 1$. Let x_n be a common fixed point of S_n and T_n [which exists by Theorem 1 of Bose and Mukherjee (1977)]. If x_0 is a cluster point of x_n, then $x_0 \in Sx_0$ and $x_0 \in Tx_0$.

PROOF : Let $x_{n_i} \to x_0$. Then

$$D(x_0, Tx_0) \leqslant d(x_0, x_{n_i}) + D(x_{n_i}, Tx_0) \leqslant d(x_0, x_{n_i}) + H(S_{n_i}x_{n_i}, Tx_0) \text{ as } x_{n_i} \in S_{n_i}x_{n_i}.$$

For simplicity in notation we write henceforward i in place of n_i.

So $$D(x_0, Tx_0) \leqslant d(x_0, x_i) + H(S_ix_i, T_ix_0) + H(T_ix_0, Tx_0)$$
$$\leqslant d(x_0, x_i) + a_i^1\, D(S_ix_i, x_i) + a_i^2\, D(T_ix_0, x_0) + a_i^3\, D(S_ix_i, x_0) + a_i^4\, D(T_ix_0, x_i) + a_i^5\, d(x_i, x_0) + H(T_ix_0, Tx_0).$$

i.e., $$\leqslant d(x_0, x_i) + a_i^2\, H(T_ix_0, Tx_0) + a_i^2\, D(Tx_0, x_0) + a_i^3\, d(x_i, x_0) + a_i^4\, H(T_ix_0, Tx_0) + a_i^4\, d(x_i, x_0) + a_i^4\, D(x_0, Tx_0) + a_i^5\, d(x_i, x_0) + H(T_ix_0, Tx_0),$$

i.e., $$(1 - a_i^2 - a_i^4)\, D(x_0, Tx_0) \leqslant (1 + a_i^3 + a_i^4 + a_i^5)\, d(x_i, x_0) + (1 + a_i^2 + a_i^4)\, H(T_i x_0, Tx_0).$$

Taking limit as $i \to \infty$, we have $(1 - a^2 - a^4)\, D(x_0, Tx_0) = 0$. But $1 - a^2 - a^4 \neq 0$, hence $x_0 \in Tx_0$ as Tx_0 is closed. Similarly $x_0 \in Sx_0$. The next theorem is a generalization to set-valued mappings of a theorem of Ray (1971) for point mappings.

Theorem 4.3 — Let $\{T_n\}$ be a sequence of multi-valued mappings from a complete bounded metric space (X, d) into $(CL(X), H)$ such that

(i) for any two multi-functions T_i, T_j, $H(T_i(x), T_j(y)) \leqslant \lambda d(x, y)$, $0 < \lambda < 1$ and $x, y \in X$ and

(ii) there is a point x_0 in X such that any two consecutive members of $\{T_n x_{n-1}\}$ are distinct where $x_n \in T_n x_{n-1}$.

Then $\{T_n\}$ has a common fixed point.

PROOF : We have $d(x_1, x_2) \leqslant \frac{1}{\mu} H(T_1 x_0, T_2 x_1)$ where $\mu = (\lambda)^{1/2}$ (μ is taken positive and it is less than one). So

$$d(x_1, x_2) \leqslant \frac{1}{\mu} H(T_1 x_0, T_2 x_1) \leqslant \mu d(x_0, x_1).$$

Also $d(x_2, x_3) \leqslant \frac{1}{\mu} H(T_2 x_1, T_3 x_2) \leqslant \mu\, d(x_1, x_2) \leqslant \mu^2 d(x_0, x_1)$.

Similarly $d(x_n, x_{n+1}) \leqslant \mu^n d(x_0, x_1)$. For $p > 0$ we have

$$\begin{aligned} d(x_n, x_{n+p}) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+p-1}, x_{n+p}) \\ &\leqslant (\mu^n + \ldots + \mu^{n+p-1})\, d(x_0, x_1) \\ &< \frac{\mu^n}{1-\mu}\, d(x_0, x_1). \end{aligned}$$

Thus $d(x_n, x_{n+p}) \to 0$ as $n \to \infty$. So $\{x_n\}$ is Cauchy in X.

Since X is complete, there is an element u in X such that $\lim_{n\to\infty} x_n = u$. For a fixed m,

$$\begin{aligned} D(u, T_m u) &\leqslant d(u, x_n) + D(x_n, T_m u) \\ &\leqslant d(u, x_n) + H(T_n x_{n-1}, T_m u) \\ &\leqslant d(u, x_n) + d(x_{n-1}, u). \end{aligned}$$

Taking limit as $n \to \infty$, we have $D(u, T_m u) = 0$. Since $T_m u$ is closed, we have $u \in T_m u$.

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