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TWO IDENTITIES DUE TO RAMANUJAN

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Proofs of two identities enunciated by Ramanujan are given.

1. INTRODUCTION

The aim of this note is to give proofs of the following two identities enunciated by Ramanujan (1957, p. 108):

(a) $$\frac{1}{1^2} + \frac{1 + \frac{1}{2} + \frac{1}{3}}{3^2} + \frac{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}}{5^2} + \&c = \frac{3}{2}\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$$

(b) $$\frac{1}{1^2} + \frac{1 + \frac{1}{3}}{2^2} + \frac{1 + \frac{1}{3} + \frac{1}{5}}{3^2} + \&c = 2\left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \&c\right)$$

It may be noted that in Ramanujan's statement of (a), there is a slip, viz., in the numerator of the third term on the left, $\frac{1}{4}$ is missing.

2. PROOFS OF (a) AND (b)

Let ζ denote the Riemann zeta function defined by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for $s > 1$. Then clearly

(c) $$\sum_{n=1}^{\infty} (2n-1)^{-3} = \frac{7}{8}\,\zeta(3).$$

Now writing $a_n = \sum_{r=1}^{n} r^{-1}$ and $b_n = a_{2n} - a_n$ for $n \geqslant 1$, it is known (Rao and Sarma 1979) that

(d) $$\sum_{n=1}^{\infty} a_n n^{-2} = 2\,\zeta(3)$$

(e) $\sum_{n=1}^{\infty} b_n n^{-2} = \frac{3}{4}\zeta(3)$.

Hence

$$\sum_{n=1}^{\infty} \frac{a_{2n-1}}{(2n-1)^2} = \sum_{n=1}^{\infty} \frac{a_n}{n^2} - \sum_{n=1}^{\infty} \frac{a_{2n}}{(2n)^2}$$

$$= \frac{3}{4} \sum_{n=1}^{\infty} \frac{a_n}{n^2} - \frac{1}{4} \sum_{n=1}^{\infty} \frac{b_n}{n^2} = \frac{21}{16}\zeta(3)$$

so that (a) follows from (c). Further

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{r=1}^{n} \frac{1}{(2r-1)} = \sum_{n=1}^{\infty} \frac{a_{2n} - \frac{1}{2} a_n}{n^2}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{a_n}{n^2} + \sum_{n=1}^{\infty} \frac{b_n}{n^2} = \frac{7}{4}\zeta(3)$$

so that again (b) follows from (c).

REFERENCES

Ramanujan, S. (1957). Note Books, Vol. 2. Tata Institute of Fundamental Research, Bombay.

Rao, R. Sita Rama Chandra, and Sarma, A. Siva Rama (1979). Some identities involving the Riemann zeta function. *Indian J. pure appl. Math.*, **10**, 602–607.