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TRANSFORMATION FORMULAE FOR THE TERMINATING GENERALIZED KAMPÉ DE FÉRIET FUNCTION

S. SARAN

Department of Mathematics, Punjabi University, Patiala 147001

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Three general transformation formulae for the generalized Kampé de Fériet function with unit arguments have been obtained. Many interesting transformations and reducible cases and sums other than those obtained by the author (Saran 1979) elsewhere have been obtained. These include as a special case a transformation of Singal (1971).

1. INTRODUCTION

Let $F^{p:m;n}_{q:r;s}$ denote the terminating generalized Kampé de Fériet function with unit arguments (Burchnall and Chaundy 1941) in the modified notation defined by

$$F^{p:m;n}_{q:r;s}\left[\begin{matrix}-N, \{A_{p-1}\} : \{a_m\}; \{b_n\} \\ \{B_q\} : \{c_r\} ; \{d_s\}\end{matrix}\right]$$

$$= \sum_{i+j\leqslant N} \frac{(-N)_{i+j} \prod\limits_{k=1}^{p-1} (A_k)_{i+j} \prod\limits_{k=1}^{m} (a_k)_i \prod\limits_{k=1}^{n} (b_k)_j}{\prod\limits_{k=1}^{q} (B_k)_{i+j} \prod\limits_{k=1}^{r} (c_k)_i \prod\limits_{k=1}^{s} (d_k)_j \, i! \, j!} \quad \ldots(1.1)$$

where $(\lambda)_n$ is defined by

$$(\lambda)_n = \begin{cases} \dfrac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \lambda(\lambda+1) \ldots (\lambda + n - 1), \; n \neq 0 \\ 1, \text{ when } n = 0. \end{cases}$$

Also, let ${}_pF_q\left[\begin{matrix}\{a_p\}; \\ \{b_q\}\end{matrix}\right]$ denote the usual hypergeometric function of unit argument defined by

$${}_pF_q\left[\begin{matrix}\{a_p\}; \\ \{b_q\}\end{matrix}\right] = \sum_{i=0}^{\infty} \frac{\prod\limits_{k=1}^{p} (a_k)_i}{\prod\limits_{k=1}^{q} (b_k)_i}, \quad \ldots(1.2)$$

where, for convergence, $\sum_1^q b_k - \sum_1^p a_k > 0$. We have assumed that the parameters may be real or complex.

2. General Transformations

The general transformations to be proved are

$$F^{1:3;r}_{1:2;s}\left[\begin{matrix} -N : a, & b, & c; A_r \\ G : & 1+a-b, 1+a-c; & B_s \end{matrix}\right] = \frac{(G-a)_N}{(G)_N}$$

$$\times \sum_{m=0}^{N} \frac{(\frac{1}{2}a)_m(\frac{1}{2}+\frac{1}{2}a)_m(1+a-b-c)_m(1+a-G)_m(-N)_m}{m!(1+a-b)_m(1+a-c)_m\{\frac{1}{2}(1+a-G-N)\}_m\{1+\frac{1}{2}(a-G-N)\}_m}$$

$$\times {}_{r+1}F_{s+1}\left[\begin{matrix} A_r, -N+m; \\ B_s, G-a-m \end{matrix}\right], \qquad \ldots(2.1)$$

$$F^{1:4;r}_{1:3;s}\left[\begin{matrix} -N : a, 1+\frac{1}{2}a & b, A_r \\ G : & \frac{1}{2}a, 1+a-b, B_s \end{matrix}\right]$$

$$= \frac{(G-a-N-1)(G-a)_{N-1}}{(G)_N}$$

$$\times \sum_{m=0}^{N} \frac{(1+\frac{1}{2}a)_m(\frac{1}{2}+\frac{1}{2}a)_m(1+a-b-c)_m(1+a-G)_m(-N)_m}{m!(1+a-b)_m(1+a-c)_m\{\frac{1}{2}(3+a-G-N)\}_m\{1+\frac{1}{2}(a-G-N)\}_m}$$

$$\times {}_{r+2}F_{s+2}\left[\begin{matrix} A_r, \frac{1}{2}(G-a-N+1), -N+m; \\ B_s, \frac{1}{2}(G-a-N-1), G-a-m \end{matrix}\right], \qquad \ldots(2.2)$$

and

$$F^{1:4;r}_{1:3;s}\left[\begin{matrix} -N: & a, & b, & c, & d; A_r \\ 2a-2k-N: & 1+a-b, & 1+a-c, & 1+a-d; & B_s \end{matrix}\right]$$

$$= \frac{(k-a)_N(1+2k-a)_{N-1}(2k-a+2N)}{(1+k)_N(2k-2a)_N}$$

$$\times \sum_{m=0}^{N} \frac{(k)_m(1+\frac{1}{2}k)_m(\frac{1}{2}a)_m(\frac{1}{2}+\frac{1}{2}a)_m(k+b-a)_m(k+c-a)_m(k+d-a)_m}{m!(\frac{1}{2}k)_m(\frac{1}{2}+k-\frac{1}{2}a)_m(1+k-\frac{1}{2}a)_m(1+a-b)_m(1+a-c)_m(1+a-d)_m}$$

$$\times {}_{r+3}F_{s+3}\left[\begin{matrix} A_r, & 1+\frac{1}{2}a-k-N, & -k-N-m & -N+m \\ B_s, & \frac{1}{2}a-k-N, & a-k-N+m, & a-2k-N-m \end{matrix}\right] \qquad \ldots(2.3)$$

where $k = 1 + 2a - b - c - d$.

The proofs of these transformations are simple and straightforward. The given function has been expressed in terms of a series involving hypergeometric series and a known transformation of 'nearly poised' series of second kind are applied. The order of the series is then inverted to obtain the formula in the final form. There are only three transformations of this type (Bailey 1964) and as such only three transformations of this type can be obtained. In fact, we have only two transformations [Bailey 1964, 4.5(1) and 4.5(2)] and the third transformation is given only to show that a formula can be obtained if G is also given a particular value.

To prove (2.1) we write

$$F^{1:3;r}_{1:2;s} = \sum_{m=0}^{N} \frac{(A_r)_m\,(-N)_m}{m!\,(G)_m\,(B_s)_m}\, {}_4F_3\left[\begin{matrix} a, & b, & c, -N+m; \\ & 1+a-b,\ 1+a-c, & G+m \end{matrix}\right].$$

Using Whipple formula (Bailey 1964, 4.5(1))

$${}_4F_3\left[\begin{matrix} a, & b & c, -m; \\ & 1+a-b,\ 1+a-c, & w \end{matrix}\right]$$

$$= \frac{(w-a)_m}{(w)_m}\, {}_5F_4\left[\begin{matrix} 1+a-w, \tfrac{1}{2}a, \tfrac{1}{2}+\tfrac{1}{2}a, 1+a-b-c, -m; \\ 1+a-b, 1+a-c, \tfrac{1}{2}(1+a-w-m), 1+\tfrac{1}{2}(a-w-m) \end{matrix}\right] \quad \ldots\text{(A)}$$

and

$$(A)_{n-m} = \frac{(-)^m\,(A)_n}{(1-A-n)_m} \quad \ldots\text{(B)}$$

and interchanging the order of summation we prove the result.

The proofs of (2.2) and (2.3) run on similar lines where we use 4.5(2) and 4.5(3) of Bailey (1964).

3. Particular Cases

A large number of particular cases of these formulae can be obtained if we use suitably the known transformations [Bailey 1964, 4.3(2), 4.3(4), 4.5(1), 4.5(2), 4.6(1) and 4.6(2)]. To quote a few, we have from (2.1) on using 4.5(1) of Bailey (1964)

$$F^{1:3;3}_{1:2;2}\left[\begin{matrix} -N: a, & b, & c; A, & B, & D \\ G: & 1+a-b,\ 1+a-c; & & k-B, & k-D \end{matrix}\right]$$

$$= \frac{(G-a)_N\,(1-G+a)_N\,(1+a-G+B+D)_N}{(G)_N\,(1-G+a+B)_N\,(1-G+a+D)_N} \times$$

(*equation continued on p.* 1144)

$$\times F^{1:4;5}_{1:3;4}\left[\begin{matrix} -N: \tfrac{1}{2}k-\tfrac{1}{2}A, \tfrac{1}{2}+\tfrac{1}{2}k-\tfrac{1}{2}A, B, D; \\ 1-G+a+B+D: \quad k-A, \tfrac{1}{2}k, \quad \tfrac{1}{2}+\tfrac{1}{2}k; \end{matrix}\right.$$

$$\left.\begin{matrix} \tfrac{1}{2}a, \tfrac{1}{2}+\tfrac{1}{2}a, 1+a-b-c, 1-G+a+B, 1-G+a+D \\ 1+a-b, 1+a-c, \tfrac{1}{2}(1+a-G-N), 1+\tfrac{1}{2}(a-G-N) \end{matrix}\right] \quad \ldots(3.1)$$

Similarly, if we use one of the transformations, say 4.3(4) of Bailey (1964), in (2.2) we obtain

$$F^{1:4;5}_{1:3;4}\left[\begin{matrix} -N: a, 1+\tfrac{1}{2}a, \quad b, \quad c; \\ 1+a+A+N: \quad \tfrac{1}{2}a, 1+a-b, 1+a-c; \end{matrix}\right.$$

$$\left.\begin{matrix} A, \quad B, \quad D, \quad E, \quad F \\ 1+A-B, 1+A-D, 1+A-E, 1+A-F \end{matrix}\right]$$

$$= \frac{(1+A)_N (1+A-E-F)_N}{(1+A-E)_N (1+A-F)_N}$$

$$\times F^{1:5;3}_{1:4;2}\left[\begin{matrix} -N: \quad 1+\tfrac{1}{2}a, \tfrac{1}{2}+\tfrac{1}{2}a, 1+a-b-c, \\ E+F-A-N: 1+a-b, 1+a-c, \end{matrix}\right.$$

$$\left.\begin{matrix} E-A-N, \quad F-A-N; E, \quad F, 1+A-B-D \\ 1-\tfrac{1}{2}A-N, \tfrac{1}{2}-\tfrac{1}{2}A-N; 1+A-D, 1+A-E \end{matrix}\right] \quad \ldots(3.2)$$

valid if $\text{Re}\,(1+A-E-F)>0$.

Similar transformations can be obtained from (2.3) also. The result of Singal (1971) can be obtained from (2.1) on using (A).

4. Reducible Cases

Many interesting reducible cases can be given from the general formulae by using the sums instead of transformations of the generalized hypergeometric function. For example, using Bailey formula [1964, 4.5(1.3)] in (2.1) by taking $r=3$, $s=2$, $A_r = A,\ 1+\tfrac{1}{2}A,\ B,\ B_s = \tfrac{1}{2}A,\ 1+A-B$, $C=\tfrac{1}{2}+\tfrac{1}{2}a$ and $G = 1+2B-N$ we get

$$F^{1:3;3}_{1:2;2}\left[\begin{matrix} -N: a, 1+\tfrac{1}{2}a, \quad b; A, 1+\tfrac{1}{2}A, \quad B \\ 1+2B-N: \quad \tfrac{1}{2}a, 1+a-b; \quad \tfrac{1}{2}A, 1+A-B \end{matrix}\right]$$

$$= \frac{(A-2B)_N (-B)_N}{(1+A-B)_N (-2B)_N}$$

$$\times {}_5F_4\left[\begin{matrix} a, 1+\tfrac{1}{2}a, \quad b, B-A-N \quad -N; \\ \tfrac{1}{2}a, 1+a-b, 1+B-N, 1-A+2B-N \end{matrix}\right]. \quad \ldots(4.1)$$

Taking $B = \frac{1}{2}(a + A) + N$ we get

$$F^{1:3;3}_{1:2;2}\left[\begin{matrix} -N: a, 1+\frac{1}{2}a, & b; A, 1+\frac{1}{2}A & \frac{1}{2}(A+a)+N \\ 1+a+A+N: & \frac{1}{2}a, 1+a-b; & \frac{1}{2}A, 1+\frac{1}{2}(A-a)-N \end{matrix}\right]$$

$$= \frac{(1+a)_{2N}(1+\frac{1}{2}a+\frac{1}{2}A)_N(1+a+A)_N}{(1+a)_N(\frac{1}{2}a-\frac{1}{2}A)_N(1+a+A)_{2N}}$$

$$\times {}_5F_4\left[\begin{matrix} a, 1+\frac{1}{2}a, & b, & \frac{1}{2}a-\frac{1}{2}A, & -N; \\ & \frac{1}{2}a, 1+a-b, & 1+\frac{1}{2}a+\frac{1}{2}A, & 1+a+N \end{matrix}\right]. \qquad \ldots(4.2)$$

This ${}_5F_4$ is well-poised and can be summed by 4.3(3) of Bailey (1964). We thus get the L.H.S. of (4.2)

$$= \frac{(1+a)_{2N}(1+a+A)_N(1+\frac{1}{2}a+\frac{1}{2}A-b)_N}{(1+a+A)_{2N}(1+a-b)_N(\frac{1}{2}a-\frac{1}{2}A)_N}. \qquad \ldots(4.1.1)$$

Also, from (2.2) if we take $G = 1 + a + A + N$, $A_r = A, D, E$ and

$$B_s = 1 + A - D, 1 + A - E$$

we get

$$F^{1:4;3}_{1:3;2}\left[\begin{matrix} -N: a, 1+\frac{1}{2}a, & b & c; A, & B, & D \\ 1+a+A+N: & \frac{1}{2}a, 1+a-b, & 1+a-c; & 1+A-B, & 1+A-D \end{matrix}\right]$$

$$= \frac{A(1+A+N)_{N-1}}{(1+a+A+N)_N}$$

$$\times \sum_{m=0}^{N} \frac{(1+\frac{1}{2}a)_m(\frac{1}{2}+\frac{1}{2}a)_m(1+a-b-c)_m(-A-N)_m(-N)_m}{m!\,(1+a-b)_m(1+a-c)_m(1-\frac{1}{2}A-N)_m(\frac{1}{2}-\frac{1}{2}A-N)_m}$$

$$\times {}_5F_4\left[\begin{matrix} A, 1+\frac{1}{2}A, & B, & D, & -N+m; \\ & \frac{1}{2}A, 1+A-B, & 1+A-D, & 1+A+N-m \end{matrix}\right].$$

This, on using Dougall formula [Bailey 1964, 4.3(3)], gives

$$\frac{A(1+A+N)_{N-1}(1+A)_N(1+A-B-D)_N}{(1+a+A+N)_N(1+A-B)_N(1+A-D)_N}$$

$$\times {}_6F_5\left[\begin{matrix} 1+\frac{1}{2}a, \frac{1}{2}+\frac{1}{2}a, 1+a-b-c, B-A-N, D-A-N, -N; \\ 1+a-b, 1+a-c, 1-\frac{1}{2}A-N, \frac{1}{2}-\frac{1}{2}A-N, B+D-A-N \end{matrix}\right]. \qquad \ldots(4.3)$$

This ${}_6F_5$ is Saalschützian and can be summed only if it is reduced to ${}_3F_2$.

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