Indian J. pure appl. Math., **11**(9) : 1147–1154, September 1980

A BIORTHOGONAL PAIR OF POLYNOMIAL SETS SUGGESTED BY A CLASS OF JACOBI POLYNOMIALS

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(*Received* 8 *May* 1980)

The problem of finding the biorthogonal pair of polynomial sets suggested by the Jacobi polynomials is partially solved. Polynomial sets $\{U_n^\alpha(x; k)\}$ and $\{V_n^\alpha(x; k)\}$ are introduced and shown to form a biorthogonal pair over the interval $(-1, 1)$ and with respect to the weight function

$$\left(\frac{1-x}{2}\right)^\alpha ; \left(\frac{1-x}{2}\right) \text{ and } \left(\frac{1-x}{2}\right)^k$$

being the basic polynomials.

1. INTRODUCTION

Konhauser (1965) extended the notion of orthogonal polynomials to two sets of polynomials and defined biorthogonality of a pair of polynomials in the following manner. Let $R_m(x)$ and $S_n(x)$ be two polynomials of degrees m and n in the basic polynomials $r(x)$ and $s(x)$ respectively. Then the sets $\{R_m(x)\}$ and $\{S_n(x)\}$ are said to be biorthogonal over the interval (a, b) with respect to the weight function $p(x)$ if

$$\int_a^b p(x)\, R_m(x)\, S_n(x)\, dx = \begin{cases} 0, m, n = 0, 1, 2, \ldots ; m \neq n \\ \neq 0, m = n. \end{cases} \qquad \ldots(1.1)$$

He also introduced (Konhauser 1967) a pair of polynomial sets $\{Y_n^\alpha(x; k)\}$ and $\{Z_n^\alpha(x; k)\}$ which are biorthogonal over the interval $(0, \infty)$ with respect to the weight function $e^{-x}x^\alpha$ of the generalized Laguerre polynomial $L_n^{(\alpha)}(x)$; the polynomial $Y_n^\alpha(x; k)$ is of degree n in x while $Z_n^\alpha(x; k)$ is of degree n in x^k, k being a positive integer. Moreover for $k = 1$, both $Y_n^\alpha(x; k)$ and $Z_n^\alpha(x; k)$ reduce to $L_n^{(\alpha)}(x)$ and the biorthogonality conditions degenerate to orthogonality requirement for $L_n^{(\alpha)}(x)$. In this sense the biorthogonal pair $\{Y_n^\alpha(x; k)\}$, $\{Z_n^\alpha(x; k)\}$ has been described to have been suggested by the orthogonal set $\{L_n^{(\alpha)}(x)\}$. Indeed this pair was studied by

Konhauser himself, Carlitz (1968), Prabhakar (1970, 1971), Srivastava (1973) and others.

Recently Prabhakar and Tomar (1979) introduced a biorthogonal pair $\{U_n(x; k)\}$, $\{V_n(x; k)\}$ of polynomial sets which is analogously suggested by the orthogonal set of Legendre polynomials $P_n(x)$, whereas the more general problem of finding the biorthogonal pair suggested by the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ remains open. In this paper, we partially solve this problem in so far as we introduce a pair $\{U_n^\alpha(x; k)\}$, $\{V_n^\alpha(x; k)\}$ which is analogously suggested by $P_n^{(\alpha,\beta)}(x)$ with $\beta = 0$. To be more explicit $U_n^\alpha(x; k)$ is a polynomial of degree n in the basic polynomial $\left(\frac{1-x}{2}\right)$ and $V_n^\alpha(x; k)$ is a polynomial of degree n in the basic polynomial $\left(\frac{1-x}{2}\right)^k$, k being a positive integer. We prove that $U_n^\alpha(x; k)$, $V_n^\alpha(x; k)$ satisfy the biorthogonality requirement with respect to the weight function and the interval of $P_n^{(\alpha,0)}(x)$. Also for $k = 1$, both $U_n^\alpha(x: k)$ and $V_n^\alpha(x; k)$ reduce to $P_n^{(\alpha,0)}(x)$. We also give some recurrence relations for both the polynomials.

2. Polynomials $U_n^\alpha(x; k)$, $V_n^\alpha(x; k)$ and Their Biorthogonality

We define the polynomial $U_n^\alpha(x; k)$ of degree n in the basic polynomial $(1 - x)/2$ by

$$U_n^\alpha(x; k) = \frac{1}{(1/k)_n} \sum_{j=0}^{n} \frac{(-n)_j}{j!} \left(\frac{1+\alpha+j}{k}\right)_n \left(\frac{1-x}{2}\right)^j \qquad \ldots(2.1)$$

and the polynomial $V_n^\alpha(x; k)$ of degree n in the basic polynomial $\left(\frac{1-x}{2}\right)^k$ by

$$V_n^\alpha(x; k) = \frac{1}{n!} \sum_{j=0}^{n} \frac{(-n)_j}{j!} (1+\alpha+kj)_n \left(\frac{1-x}{2}\right)^{kj} \qquad \ldots(2.2)$$

where k is a positive integer. For $k = 1$, both $U_n^\alpha(x; k)$ and $V_n^\alpha(x; k)$ reduce to $P_n^{(\alpha,0)}(x)$ [Rainville 1960, 132(1)],

$$P_n^{(\alpha,0)}(x) = \frac{(1+\alpha)_n}{n!} \, {}_2F_1\left[\begin{matrix} -n,\ 1+\alpha+n; \\ 1+\alpha \quad ; \end{matrix} \ \frac{1-x}{2}\right]. \qquad \ldots(2.3)$$

Moreover, for $\alpha = 0$, the pair $\{U_n^{\alpha}(x;k)\}$, $\{V_n^{\alpha}(x;k)\}$ becomes the biorthogonal pair $\{U_n(x;k)\}$, $\{V_n(x;k)\}$ considered by Prabhakar and Tomar (1979). Indeed when $k = 1$ and $\alpha = 0$, both $U_n^{\alpha}(x;k)$ and $V_n^{\alpha}(x;k)$ reduce to $P_n(x)$. It is not difficult to see that $V_n^{\alpha}(x;k)$ has a hypergeometric form

$$V_n^{\alpha}(x;k) = \frac{(1+\alpha)_n}{n!}\ {}_{k+1}F_k\left[\begin{matrix} -n,\ \triangle(k, 1+\alpha+n); \\ \triangle(k, 1+\alpha)\quad ; \end{matrix}\ \left(\frac{1-x}{2}\right)^k\right] \quad \text{...(2.4)}$$

where $\triangle(k, n)$ denotes the set of k parameters

$$\frac{n}{k}, \frac{n+1}{k}, \frac{n+2}{k}, \ldots, \frac{n+k-1}{k}.$$

To show that these polynomials are biorthogonal, it is sufficient (Konhauser 1965) to show that

$$\int_{-1}^{1} \left(\frac{1-x}{2}\right)^{\alpha+ki} U_n^{\alpha}(x;k)\,dx = \begin{cases} 0, & i = 0, 1, 2, \ldots, (n-1) \\ \neq 0 & i = n \end{cases} \quad \text{...(2.5)}$$

and

$$\int_{-1}^{1} \left(\frac{1-x}{2}\right)^{\alpha+i} V_n^{\alpha}(x;k)\,dx = \begin{cases} 0, & i = 0, 1, 2, \ldots, (n-1) \\ \neq 0, & i = n. \end{cases} \quad \text{...(2.6)}$$

In fact, (2.5) and (2.6) together are equivalent to the biorthogonality requirement (1.1).

To establish (2.5), let

$$I_{i,n} = \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{\alpha+ki} U_n^{\alpha}(x;k)\,dx$$

$$= \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{\alpha+ki} \frac{1}{(1/k)_n} \sum_{j=0}^{n} \frac{(-n)_j}{j!} \left(\frac{1+\alpha+j}{k}\right)_n \left(\frac{1-x}{2}\right)^j dx$$

$$= \frac{1}{(1/k)_n} \sum_{j=0}^{n} \frac{(-n)_j}{j!} \left(\frac{1+\alpha+j}{k}\right)_n \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{\alpha+ki+j} dx$$

$$= \frac{2}{(1/k)_n} \sum_{j=0}^{n} \frac{(-n)_j}{j!} \left(\frac{1+\alpha+j}{k}\right)_n \frac{1}{\alpha+ki+j+1}. \quad \text{...(2.7)}$$

Thus for $i < n$,

$$I_{i,n} = \frac{2}{k(1/k)_n} \sum_{j=0}^{n} \frac{(-n)_j}{j!} \left(\frac{1+\alpha+j}{k} + i + 1\right)_{n-i-1} \left(\frac{1+\alpha+j}{k}\right)_i .$$

The right-hand side being the nth difference of a polynomial of degree $(n-1)$ in $1/k$ is zero. Hence we have

$$I_{i,n} = 0, \quad i < n.$$

Also for $i = n$, we have

$$I_{n,n} = \frac{2}{(1/k)_n} \sum_{j=0}^{n} \frac{(-n)_j}{j!} \left(\frac{1+\alpha+j}{k}\right)_n \frac{1}{\alpha+j+kn+1}. \qquad \text{...(2.8)}$$

Since $\left(\frac{1+\alpha+j}{k}\right)_n$ is a polynomial of degree n in j, $\left(\frac{1+\alpha+j}{k} + n\right)$ is a factor of $\left[\left(\frac{1+\alpha+j}{k}\right)_n - (-n)_n\right]$, so that we can write

$$\left(\frac{1+\alpha+j}{k}\right)_n - (-n)_n = \left(\frac{1+\alpha+j}{k} + n\right) A_{n-1}(j) \qquad \text{...(2.9)}$$

where $A_{n-1}(j)$ is a polynomial of degree $(n-1)$ in j. Hence we have

$$I_{n,n} = \frac{2}{(1/k)_n} \sum_{j=0}^{n} \frac{(-n)_j}{j!} \frac{1}{\alpha+j+kn+1} \times \left[\left(\frac{1+\alpha+j}{k} + n\right) A_{n-1}(j) + (-n)_n\right]$$

$$= \frac{2}{k(1/k)_n} \sum_{j=0}^{n} \frac{(-n)_j}{j!} A_{n-1}(j) + \frac{2}{(1/k)_n} \sum_{j=0}^{n} \frac{(-n)_j}{j!} \frac{(-n)_n}{(\alpha+j+kn+1)}.$$

The first sum on the right-hand side is evidently zero and

$$I_{n,n} = \frac{2(-n)_n}{(1/k)_n} \sum_{j=0}^{n} \frac{(-n)_j}{j!} \frac{1}{(\alpha+j+kn+1)}$$

$$= \frac{2(-n)_n \, n!}{(1/k)_n \, (\alpha+kn+1)_{n+1}} \qquad \text{...(2.10)}$$

using the identity

$$\sum_{j=0}^{n} \frac{(-n)_j}{j!} \frac{1}{a+j} = \frac{n!}{(a)_{n+1}}. \qquad \text{...(2.11)}$$

Thus we have shown that

$$I_{n,n} = \frac{2(-n)_n\, n!}{(1/k)_n\,(\alpha + kn + 1)_{n+1}}. \qquad ...(2.12)$$

To establish (2.6), let

$$J_{n,i} = \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{\alpha+i} V_n^{\alpha}(x;k)\, dx$$

$$= \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{\alpha+i} \frac{1}{n!} \sum_{j=0}^{n} \frac{(-n)_j}{j!} (1+\alpha+kj)_n \left(\frac{1-x}{2}\right)^{kj} dx$$

$$= \frac{1}{n!} \sum_{j=0}^{n} \frac{(-n)_j}{j!} (1+\alpha+kj)_n \int_{-1}^{1} \left(\frac{1-x}{2}\right)^{\alpha+i+kj} dx$$

$$= \frac{2}{n!} \sum_{j=0}^{n} \frac{(-n)_j}{j!} (1+\alpha+kj)_i\, (2+\alpha+kj+i)_{n-i-1} \qquad ...(2.13)$$

Here also the right-hand side is zero when $i < n$ and we get

$$J_{n,n} = 0, \quad i < n.$$

And for $i = n$, we have

$$J_{n,n} = \frac{2}{n!} \sum_{j=0}^{n} \frac{(-n)_j}{j!} (1+\alpha+kj)_n \frac{1}{1+\alpha+n+kj}$$

$$= \frac{2}{n!} \sum_{j=0}^{n} \frac{(-n)_j}{j!} \frac{1}{1+\alpha+n+kj}$$

$$\times \left[(1+\alpha+kj+n)\, B_{n-1}(j) + (-n)_n\right]$$

where $B_{n-1}(j)$ is a polynomial of degree $(n-1)$ in j. So

$$J_{n,n} = \frac{2(-1)^n}{k} \sum_{j=0}^{n} \frac{(-n)_j}{j!} \left(\frac{1+\alpha+n}{k} + j\right)^{-1}$$

$$= \frac{2(-1)^n}{k}\, n! \left(\left(\frac{1+\alpha+n}{k}\right)_{n+1}\right)^{-1}, \qquad ...(2.14)$$

using the identity (2.11). For $k = 1$, both (2.12) and (2.14) lead to

$$\int_{-1}^{1} \left(\frac{1-x}{2}\right)^{\alpha+n} P_n^{(\alpha,0)}(x)\,dx = \frac{2(-1)^n\, n!}{(1+\alpha+n)_{n+1}}. \qquad \ldots(2.15)$$

Further for $\alpha = 0$, we get the following integral formula for Legendre polynomials

$$\int_{-1}^{1} \left(\frac{1-x}{2}\right)^{n} P_n(x)\,dx = \frac{2(-1)^n\,(n!)^2}{(2n+1)!}.$$

3. Pure Recurrence Relations

Konhauser (1965) gave a general method for finding pure recurrence relations for the polynomials forming a biorthogonal pair. It follows from Theorem 2.6 of Konhauser (1965) that there are pure recurrence relations of the form

$$\left(\frac{1-x}{2}\right)^{k} U_n^{\alpha}(x;k) = \sum_{i=n-1}^{n+k} a_{n,i} U_i^{\alpha}(x;k) \qquad \ldots(3.1)$$

and

$$\left(\frac{1-x}{2}\right)^{k} V_n^{\alpha}(x;k) = \sum_{i=n-k}^{n+1} b_{n,i} V_i^{\alpha}(x;k) \qquad \ldots(3.2)$$

each connecting $(k+2)$ successive polynomials.

The coefficients $a_{n,i}$ and $b_{n,i}$ are functions of n and not of x and are computed as follows. Write $U_n^{\alpha}(x;k)$ as

$$U_n^{\alpha}(x;k) = \sum_{j=0}^{n} C_{n,j} \left(\frac{1-x}{2}\right)^{j} \qquad \ldots(3.3)$$

where

$$C_{n,j} = \frac{(-1)^j \binom{n}{j}}{(1/k)_n} \left(\frac{1+\alpha+j}{k}\right)_n.$$

Substituting for $U_n^{\alpha}(x;k)$ from (3.3) in (3.1), we get

$$\left(\frac{1-x}{2}\right)^{k} \sum_{j=0}^{n} C_{n,j} \left(\frac{1-x}{2}\right)^{j} = \sum_{j=n-1}^{n+k} a_{n,j} \sum_{i=0}^{j} C_{j,i} \left(\frac{1-x}{2}\right)^{i}. \qquad \ldots(3.4)$$

On comparing the coefficients of $\left(\frac{1-x}{2}\right)^{n+k}, \left(\frac{1-x}{2}\right)^{n+k-1}, \left(\frac{1-x}{2}\right)^{n+k-2}, \ldots$, we get

$$C_{n,n} = a_{n,n+k}C_{n+k,n+k} \quad \ldots(3.5)$$

$$C_{n,n-1} = a_{n,n+k}C_{n+k,n+k-1} + a_{n,n+k-1}C_{n+k-1,n+k-1} \quad \ldots(3.6)$$

$$C_{n,n-2} = a_{n,n+k}C_{n+k,n+k-2} + a_{n,n+k-1}C_{n+k-1,n+k-2} + a_{n,n+k-2}C_{n+k-2,n+k-2}$$

$$\ldots \quad \ldots \quad \ldots$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots(3.7)$$

In the particular case $k = 1$, we get from (3.5) - (3.7)

$$a_{n,n+1} = \frac{C_{n,n}}{C_{n+1,n+1}} = -\frac{(n+1)(1+\alpha+n)}{(1+\alpha+2n)(2+\alpha+2n)},$$

$$a_{n,n} = \frac{C_{n,n-1}}{C_{n,n}} - \frac{C_{n+1,n}}{C_{n+1,n+1}}$$

$$= \tfrac{1}{2}\left[1 + \frac{\alpha^2}{(\alpha+2n)(2+\alpha+2n)}\right]$$

$$a_{n,n-1} = \frac{C_{n,n-2}}{C_{n-1,n-1}} - \frac{C_{n,n}C_{n+1,n}}{C_{n-1,n-1}C_{n+1,n+1}} - \frac{C_{n,n-1}C_{n,n-1}}{C_{n-1,n-1}C_{n,n}} + \frac{C_{n+1,n}C_{n,n-1}}{C_{n-1,n-1}C_{n+1,n+1}}$$

$$= \frac{-n(\alpha+n)}{(\alpha+2n)(\alpha+2n+1)}$$

and (3.3) reduces to three term recurrence relation (Rainville 1960, 137(1)) for Jacobi polynomials $P_n^{(\alpha,0)}(x)$

$$2n(\alpha+n)(\alpha+2n-2)\,P_n^{(\alpha,0)}(x)$$

$$= (\alpha+2n-1)\,[\alpha^2 + x(\alpha+2n)(\alpha+2n-2)]\,P_{n-1}^{(\alpha,0)}(x)$$

$$-\,2(\alpha+n-1)(n-1)(\alpha+2n)\,P_{n-2}^{(\alpha,0)}(x). \quad \ldots(3.8)$$

The coefficients $b_{n,i}$ in the recurrence relation (3.2) can be calculated by the same method. For $k = 1$, (3.2) also reduces to the three term recurrence relation (3.8).

Remark : Several generating functions, integrals and other results on the polynomials $U_n^\alpha(x; k)$, $V_n^\alpha(x; k)$ have been obtained and will be reported elsewhere.

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