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NOTE ON "SOME FORMULAS INVOLVING THE PRODUCTS OF SEVERAL JACOBI OR LAGUERRE POLYNOMIALS"

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Several multilateral generating functions involving both the Jacobi and Laguerre polynomials are obtained by essentially specializing a general result due to Srivastava and Singhal (1972) containing the product of several generalized hypergeometric polynomials. Some alternative derivations are indicated.

§1. Srivastava and Singhal (1972) proved the following multiple series relation involving the product of several Laguerre polynomials first by mathematical induction, and then by using an operational technique [Srivastava and Singhal 1972, p. 1239, eqn. (5)] :

$$\sum_{n_1, \ldots, n_k=0}^{\infty} \frac{(m + \Sigma\, n_i)!\, L^{(\alpha)}_{m+\Sigma n_i}(x)\, L^{(\beta_1)}_{n_1}(y_1) \ldots L^{(\beta_k)}_{n_k}(y_k)\, u_1^{n_1} \ldots u_k^{u_k}}{(1+\beta_1)_{n_1} \ldots (1+\beta_k)_{n_k}}$$

$$= (1+\alpha)_m\, \theta_k^{-\alpha-m-1}\, e^x\, \psi_2^{(k+1)} \left[1+\alpha+m;\, 1+\alpha,\, 1+\beta_1, \ldots, 1+\beta_k; \frac{-x}{\theta_k}, \frac{-u_1 y_1}{\theta_k}, \ldots, \frac{-u_k y_k}{\theta_k} \right], (\alpha > -1, \beta_i > -1, i = 1, \ldots, k), \quad \ldots(1)$$

where $\psi_2^{(n)}$ is a confluent hypergeometric function of n variables [see Erdélyi *et al.* 1954, p. 445] and for convenience we have $\theta_k = 1 - \sum_1^k u_i$, $k = 1, 2, 3, \ldots$. This notation has been used throughout this note.

From (1) they obtained, by Laplace-transforming an ${}_1F_1$ function into a ${}_2F_1$ function, the following multilinear generating function involving Jacobi polynomials [Srivastava and Singhal 1972, p. 1245, eqn. (29)] :

$$\sum_{n_1, \ldots, n_k=0}^{\infty} \frac{(m + \Sigma\, n_i)!\, P^{(\alpha, \beta-\Sigma n_i)}_{m+\Sigma n_i}(x)\, P^{(\beta_1, \delta_1-n_1)}_{n_1}(y_1) \ldots P^{(\beta_k, \delta_k-n_k)}_{n_k}(y_k)\, u_1^{n_1} \ldots u_k^{n_k}}{(1+\beta_1)_{n_1} \ldots (1+\beta_k)_{n_k}} =$$

(*equation continued on p.* 1159)

$$= (1+\alpha)_m \left(\frac{1+x}{2}\right)^{-\alpha-\beta-m-1} \theta_k^{-\alpha-m-1} F_A^{(k+1)} \Big[1+\alpha+m,$$

$$1+\alpha+\beta+m;\ 1+\beta_1+\delta_1, \ldots, 1+\beta_k+\delta_k;\ 1+\alpha,$$

$$1+\beta_1, \ldots, 1+\beta_k;\ \frac{x-1}{\theta_k(x+1)}, \frac{(y_1-1)\,u_1}{2\theta_k}, \ldots, \frac{(y_k-1)\,u_k}{\theta_k} \Big] \qquad \ldots(2)$$

where $F_A^{(n)}$ is Lauricella's hypergeometric function of n variables of the first kind [see Erdélyi *et al.* 1954, p. 445], θ_k is the same as above, and

$$\alpha, \beta > -1, \alpha_i, \beta_i > -1 \text{ with } i = 1, \ldots, k.$$

Results (1) and (2) are, in fact, particular cases of a more general formula of Srivastava and Singhal (1972, p. 1244, eqn. (24)) containing the product of several generalized hypergeometric polynomials.

The purpose of this short note is to point out some interesting results that are contained in the aforementioned formula (24) of Srivastava and Singhal (1972). However, we shall alternatively obtain them from the Srivastava-Singhal result (1) by using the following (Feldheim's) integral formula

$$\Gamma(1+\alpha+\beta+n)\, P_n^{(\alpha,\beta)}(x) = \int_0^\infty t^{\alpha+\beta+n} e^{-t} L_n^{(\alpha)}\left(\frac{1-x}{2}\, t\right) dt \qquad \ldots(3)$$

where $\alpha+\beta > -1$, $n = 0, 1, 2, \ldots$. This method is substantially the same as the Laplace transform technique used earlier by Srivastava and Singhal (1972).

§2. Multiply both the sides of (1) by $t^{\alpha+\beta+m} e^{-t}$ and replace x by $\frac{1}{2}(1-x)\, t$ in (1). Integrate w.r.t. t over the interval $(0, \infty)$ to obtain on account of (3), the following mixed relation involving Jacobi as well as Laguerre polynomials :

$$\sum_{n_1, \ldots, n_k=0}^{\infty} \frac{(m+\Sigma n_i)!\, P_{m+\Sigma n_i}^{(\alpha, \beta-\Sigma n_i)}(x)\, L_{n_1}^{(\beta_1)}(y_1) \ldots L_{n_k}^{(\beta_k)}(y_k)\, u_1^{n_1} \ldots u_k^{n_k}}{(1+\beta_1)_{n_1} \ldots (1+\beta_k)_{n_k}}$$

$$= (1+\alpha)_m \left(\frac{1+x}{2}\right)^{-\alpha-\beta-m-1} \theta_k^{-\alpha-\beta-m-1}$$

$$\times F^{1:1;0;\cdots;0}_{0:1;1;\cdots;1}\left(\begin{matrix} [1+\alpha+m : 1, \ldots, 1] : [1+\alpha+\beta+m : 1]; \\ \text{—} \qquad\qquad : [1+\beta_1 \qquad : 1]; \ldots; \end{matrix} \right.$$

$$\left. \begin{matrix} \text{—} \qquad ; \\ [1+\beta_k : 1]\ ; \end{matrix}\ \frac{x-1}{(x+1)\,\theta_k}, \frac{-u_1 y_1}{\theta_k}, \ldots, \frac{-u_k y_k}{\theta_k} \right). \qquad \ldots(4)$$

On the right-hand side of (4) use of the notation for the generalized Lauricella functions of several variables defined and studied by Srivastava and Daoust (1969,

p. 454) has been made. In formulas (5) and (6) that follow this very notation has been used.

In (4) replace y_1 by $\frac{1}{2}(1 - y_1)t$ and multiply both the sides by $t^{\beta_1+\delta_1} e^{-t}$ and integrate w.r.t. t over the interval $(0, \infty)$ to obtain as a consequence of (3) the following multilateral generating function

$$\sum_{n_1,\ldots,n_k=0}^{\infty} \frac{(m+\Sigma n_i)!\, P_{m+\Sigma n_i}^{(\alpha,\beta-\Sigma n_i)}(x)\, P_{n_1}^{(\beta_1,\delta_1-n_1)}(y_1)\, L_{n_2}^{(\beta_2)}(y_2)\ldots L_{n_k}^{(\beta_k)}(y_k)\, u_1^{n_1}\ldots u_k^{n_k}}{(1+\beta_1)_{n_1}\ldots(1+\beta_k)_{n_k}}$$

$$= (1+\alpha)_m \left(\frac{1+x}{2}\right)^{-\alpha-\beta-m-1} \theta_k^{-\alpha-m-1}$$

$$\times F^{1:1;1;0;\cdots;0}_{0:1;1;1;\cdots;1}\left(\begin{matrix}[1+\alpha+m:1,1,\ldots 1]:[1+\alpha+\beta+m:1]; \\ -\quad :\quad [1+\alpha\quad :1];\end{matrix}\right.$$

$$\begin{matrix}[1+\beta_1+\delta_1:1]; & - & ; \\ [1+\beta_1\quad :1]; & [1+\beta_2:1]; &\end{matrix}$$

$$\left.\begin{matrix}\ldots; & - & ; \\ \ldots; & [1+\beta_k:1]; &\end{matrix}\ \frac{x-1}{(x+1)\theta_k},\ \frac{(y_1-1)u_1}{2\theta_k},\ \frac{-y_2u_2}{\theta_k},\ \ldots,\ \frac{-y_ku_k}{\theta_k}\right). \qquad \ldots(5)$$

If we continue in this way lastly one would obtain relation (2). All these results can be put in the form (with $1 \leqslant i \leqslant k$) :

$$\sum_{n_1,\ldots,n_k=0}^{\infty} \frac{(m+\Sigma n_i)!\, P_{m+\Sigma n_i}^{(\alpha,\beta-\Sigma n_i)}(x)\, P_{n_1}^{(\beta_1,\delta_1-n_1)}(y_1)\ldots P_{n_i}^{(\beta_i,\delta_i-n_i)}(y_i)}{(1+\beta_1)_{n_1}\ldots(1+\beta_k)_{n_k}}$$

$$\times L_{n_{1+i}}^{(\beta_{i+1})}(y_{i+1})\ldots L_{n_k}^{(\beta_k)}(y_k)\, u_1^{n_1}\ldots u_k^{n_k}$$

$$= (1+\alpha)_m \left(\frac{1+x}{2}\right)^{-1-\alpha-\beta-m} \theta_k^{-\alpha-m-1}$$

$$\times F^{1:1;\cdots;1;0;\cdots;0}_{0:1;\cdots;1;1;\cdots;1}\left(\begin{matrix}[1+\alpha+m:1;1,\ldots;1]:[1+\alpha+\beta+m:1]; \\ -\quad :\quad [1+\alpha\quad :1];\end{matrix}\right.$$

$$\begin{matrix}[1+\beta_1+\delta_1:1]; \ldots; [1+\beta_i+\delta_i:1]; & - & ; \\ [1+\beta_1\quad :1]; \quad - & ; [1+\beta_k:1]; &\end{matrix}$$

$$\left.\frac{x-1}{(x+1)\theta_k},\ \frac{(y_1-1)u_1}{2\theta_k},\ \ldots,\ \frac{(y_i-1)u_i}{2\theta_k},\ \frac{-y_{i+1}u_{i+1}}{\theta_k},\ \ldots,\ \frac{-y_ku_k}{\theta_k}\right). \qquad \ldots(6)$$

For each i, $(1 \leqslant i \leqslant k)$, one obtains from (6) several multilateral generating relations involving the Jacobi and the Laguerre polynomials.

§3. The relations (4), (5) and (6), which are obvious special cases of the general Srivastava-Singhal formula [1972, p. 1244, eqn. (24)], can alternatively be derived as limiting forms of multilinear generating function (2) also due to Srivastava and Singhal [1972, p. 1245, eqn. (29)].

For examaple, if in (2) we set $\delta_j = -1 - \beta_j + 1/\epsilon$ and $y_j \to 1 - 2\epsilon y_j$, $j = i+1, \ldots, k$ and take the limit of both sides of the resulting equation as $\epsilon \to 0$, we shall at once arrive at (6), since

$$\lim_{\epsilon \to 0} P_n^{(\alpha, 1/\epsilon)}(1 - 2\epsilon x) = L_n^{\alpha}(x), \; n \geqslant 0.$$

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