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DEGREE OF APPROXIMATION BY CESÀRO MEAN OF FOURIER-LAGUERRE SERIES

B. K. BEOHAR AND B. L. JADIYA

Department of Mathematics, Madhav Vigyan Mahavidyalaya, Ujjain 456010, *M.P.*

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In this note, we estimate the degree of approximation by Cesàro mean of order $k > \alpha > -1$, which is an extension of the results of Gupta (1971), Singh (1978) and Beohar and Jadiya (1978) in which $k > \alpha + \frac{1}{2}$.

§1. The Fourier-Laguerre expansion of a function $f(x) \in L\,[0, \infty)$ is

$$f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x),\ \alpha > -1 \qquad \ldots(1.1)$$

where

$$a_n = \left\{(\alpha + 1)\binom{n+\alpha}{\alpha}\right\}^{-1} \int_0^{\infty} e^{-y} y^{\alpha} f(y)\, L_n^{(\alpha)}(y)\, dy \qquad \ldots(1.2)$$

and $L_n^{(\alpha)}(x)$ denotes the nth Laguerre polynomial of order $\alpha > -1$.

If $\Sigma\, u_n$ be a given infinite series, the nth Cesàro mean of order k ($k > -1$) is defined as

$$\sigma_n^{(k)}(x) = \frac{S_n^{(k)}(x)}{A_n^k}. \qquad \ldots(1.3)$$

Degree of approximation by Cesàro mean of Laguerre expansion has been studied by Gupta (1971), Singh (1978) and Beohar and Jadiya (1978). In all these results the order of Cesàro mean is greater $-\frac{1}{2}$. The object of the present paper is to estimate the order of Cesàro mean of order $k > \alpha > -1$. In fact we prove the following :

Theorem — For $k > \alpha > -1$

$$\sigma_n^{(k)}(f, 0) = O(n^{-1/4}) + O\{\psi(1/n)\} \qquad \ldots(1.4)$$

provided

$$\int_0^t | df(y) | \leqq A\psi(t), 0 \leqq t \leqq \omega < \infty \quad ...(1.5)$$

$$\int_\omega^\infty e^{-y/2}y^{(6\alpha-6k-1)/12} | df(y) | < \infty \quad ...(1.6)$$

and

$$\int_\omega^\infty e^{-y/2}y^{(6\alpha-6k-13)/12} | f(y) | < \infty \quad ...(1.7)$$

where $\psi(t)$ is a positive increasing function such that

$$\int_{c/n}^{\delta} \frac{\psi(t)}{t^2} dt = O\{n\psi(1/n)\}, \quad n \to \infty. \quad ...(1.8)$$

§2. *Lemma* — If $f(x)$ and $L_n^{(\alpha)}(x)$ are defined as in section 1, then

$$e^{-y}y^{\alpha}L_n^{(\alpha+k+1)}(y) f(y) dy$$

$$= \frac{k+1}{n}\int_0^\infty e^{-y}y^{\alpha}L_{n-1}^{(\alpha+k+2)}(y) f(y) dy$$

$$- \frac{1}{n}\int_0^\infty e^{-y}y^{\alpha+1}L_{n-1}^{(\alpha+k+2)}(y) df(y). \quad ...(2.1)$$

With the help of relation

$$D[e^{-y}y^{\alpha+1}DL_n^{(\alpha)}(x)] + e^{-y}y^{\alpha}L_n^{(\alpha)}(y) = 0, D \equiv \frac{d}{dx}$$

given in Rainville (1960) and simplifying we can get easily the Lemma.

§3. *Proof of the Theorem* — As given in Szegö (1959), we have

$$\sigma_n^{(k)}(f; 0) = \{\Gamma(\alpha+1) A_n^k\}^{-1} \int_0^\infty e^{-y}y^{\alpha}L_n^{(\alpha+k+1)}(y) f(y) dy$$

$$= \int_0^{c/n} + \int_{c/n}^{\omega} + \int_\omega^\infty$$

$$= \sum_{i=1}^{3} I_i, \quad \text{say}.$$

Considering I_1, and using the Lemma, we get

$$I_1 = O(n^{-k-1}) [- \int_0^{c/n} e^{-y} y^{\alpha} L_{n-1}^{(\alpha+k+2)}(y)\, df(y)$$

$$+ (k+1) \int_0^{c/n} e^{-y} y^{\alpha} L_{n-1}^{(\alpha+k+2)}(y) f(y)\, dy.$$

Now, using the well-known result (Szegö 1959, p. 175)

$$L_n^{(\alpha)}(x) = O(n^{\alpha}), \quad \text{for } 0 < x < c/n, \alpha > -1,$$

we get

$$I_1 = O(n^{\alpha+1}) [\int_0^{c/n} e^{-y} y^{\alpha+1} \mid df(y) \mid + \int_0^{c/n} e^{-y} y^{\alpha} \mid f(y) \mid dy].$$

Now, using the fact

$$\mid f(y) \mid \leqq \int_0^{y} \mid df(y) \mid \leqq A\psi(y) \qquad \text{...(3.1)}$$

we get

$$I_1 = O(n^{\alpha+1}) [O(n^{-\alpha-1}) \int_0^{c/n} \mid df(y) \mid + \int_0^{c/n} y^{\alpha}\psi(y)\, dy]$$

$$= O(n^{\alpha+1}) [O(n^{-\alpha-1})\, \psi(1/n) + \psi(1/n) \int_0^{c/n} y^{\alpha} dy]$$

$$= O\{\psi(1/n)\}. \qquad \text{...(3.2)}$$

Again using the order estimates for Laguerre polynomials (Szegö 1959, p. 175)

$$L_n^{(\alpha)}(x) = x^{-\alpha/2-1/4} . O(n^{\alpha/2-1/4}), \text{ for } c/n \leqslant x \leqslant \omega,$$

and Lemma, we get

$$I_2 = O(n^{(2\alpha-2k-1)/4}) [\int_{c/n}^{\omega} y^{(2\alpha-2k-5)/4} \mid f(y) \mid dy$$

$$+ \int_{c/n}^{\omega} y^{(2\alpha-2k-1)/4} \mid df(y) \mid .$$

Using (3.1), we get

$$I_2 = O(n^{-1/4}) [\{y^{-1/4}\psi(y)\}_{c/n}^{\omega} + \int_{c/n}^{\omega} y^{-5/4}\psi(y)\, dy]$$

$$= O(n^{-1/4}) [O(1) + O(n^{1/4}\psi(1/n))]$$

$$= O(n^{-1/4}) + O\{\psi(1/n)\} \qquad \text{...(3.3)}$$

Finally, using the relation (Szegö 1959, p. 238), we get

$$I_3 = O(n^{-k-1}) \left[\int_{\omega}^{\infty} e^{-y/2} y^{(6\alpha+6k+13)/12} \mid L_n^{(\alpha+k+2)}(y) \mid y^{(6\alpha-6k-1)/12}\, df(y) \right.$$

$$\left. + \int_{\omega}^{\infty} e^{-y/2} y^{(6\alpha+6k+13)/12} \mid L_n^{(\alpha+k+2)}(y) \mid y^{(6\alpha-6k-13)/12} f(y)\, dy \right]$$

$$= O(n^{(-4k-4+2\alpha+2k+3)/4}) \left[\int_{\omega}^{\infty} e^{-y/2} y^{(6\alpha-6k-1)/12} df(y) \right.$$

$$\left. + \int_{\omega}^{\infty} e^{-y/2} y^{(6\alpha-6k-13)/12} f(y)\, dy \right]$$

$$= O(n^{(-2k-1+2\alpha)/4}) \,.\, O(1)$$

$$= O(n^{-1/4}) \qquad \ldots(3.4)$$

Thus by virtue of (3.2) to (3.4), the theorem is proved.

REFERENCES

Beohar, B. K., and Jadiya, B. L. (1978). Degree of approximation of a function by Cesàro mean of Laguerre expansion. *Indian J. pure appl. Math.*, **9**, 1304–11.

Gupta, D. P. (1971). Degree of approximation by Cesàro mean of Fourier-Laguerre expansion. *Acta. Sci. Math.* (*Szeged*), **32**, 255–59.

Rainville, E. D. (1960). Special Functions. MacMillan Co., New York.

Singh, T. (1978). Degree of approximation by Cesàro mean of Fourier-Laguerre expansion. *Indian J. pure appl. Math.*, **9**, 394–99.

Szegö, G. (1959). Orthogonal Polynomials. Amer. Math. Soc. Colloq. Publ., New York.