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SOME INEQUALITIES FOR *H*-FUNCTIONS—II

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In the present paper, we establish a number of inequalitles for the *H*-function with the help of certain integrals due to Gupta and Jain (1966) and known inequalities given by Luke (1972). Inequalities recently given by Raina and Koul (1976) have been shown as particular cases of these results.

1. INTRODUCTION

Recently, Luke (1972) obtained a number of inequalities for the generalized hypergeometric function. Using some of these known inequalities, Raina and Koul (1976) derived certain inequalities for the Fox's *H*-function. Singh (1978) also established some inequalities for Fox's *H*-function by starting with different relations given by Luke (1972). This paper which is in continuation of the work of Singh (1978), generalizes the results given by Raina and Koul.

The *H*-function introduced by Fox (1961, p. 408) will be represented as follows:

$$H_{r,s}^{m,n}\left[z\,\middle|\,\begin{matrix}((a_r, A_r))\\((b_s, B_s))\end{matrix}\right]$$

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - B_j u)\prod_{j=1}^{n}\Gamma(1 - a_j + A_j u)}{\prod_{j=m+1}^{s}\Gamma(1 - b_j + B_j u)\prod_{j=n+1}^{r}\Gamma(a_j - A_j u)}\, z^u du \quad ...(1.1)$$

where $z \neq 0$, m, n, r, s are integers satisfying $1 \leqslant m \leqslant s$, $0 \leqslant n \leqslant r$ and $((a_r, A_r))$ represents the set of parameters $(a_1, A_1), (a_2, A_2), \ldots, (a_r, A_r)$.

Asymptotic expansion and analytic continuation of the *H*-function have been discussed by Braaksma (1963).

The integral in (1.1) converges when $|\arg z| < \frac{1}{2}\mu\pi$, where

$$\mu = \sum_1^n (A_j) - \sum_{n+1}^r (A_j) + \sum_1^m (B_j) - \sum_{m+1}^s (B_j) > 0. \quad ...(1.2)$$

Because of large number of parameters $(a_j, A_j)_{n+1,r}$ has been used to denote the set of parameters $(a_{n+1}, A_{n+1}), (a_{n+2}, A_{n+2}), \ldots, (a_r, A_r)$.

2. Inequalities for H-Functions

The following inequalities are established for real numbers with $z > 0$, $\delta > 0$, $\beta_j \geqslant \alpha_j \geqslant 0$ $(j = 1, 2, \ldots, p)$:

$$(\sigma\theta)^{-\lambda} H^{m+1,n+1}_{r+1,s+1}\left[z\,\middle|\,\begin{matrix}((a_n, A_n)), (1-\lambda, \delta), (a_j, A_j)_{n+1,r}\\ ((b_m, B_m)), (1-\lambda, \delta)\ (b_j, B_j)_{m+1,s}\end{matrix}\right]$$

$$< \frac{\psi}{\Gamma(\sigma)} H^{m+p+1,n+1}_{r+p+1,s+p+1}\left[z\,\middle|\,\begin{matrix}((a_n, A_n)), (1-\lambda, \delta), ((\beta_p-\lambda, \delta)), (a_j, A_j)_{n+1,r}\\ ((b_m, B_m)), (\sigma-\lambda, \delta), ((\alpha_p-\lambda, \delta)), (b_j, B_j)_{m+1,s}\end{matrix}\right]$$

$$< \left\{1 - \frac{2\sigma\theta}{(\sigma+1)\varphi}\right\} M_1 \delta^{-1} z^{-\nu} + \sigma\theta \left\{\frac{(\sigma+1)\varphi}{2}\right\}^{-(\lambda+1)}$$

$$\times H^{m+1,n+1}_{r+1,s+1}\left[z\left\{\frac{\sigma+1)\varphi}{2}\right\}^{-\delta}\,\middle|\,\begin{matrix}((a_n, A_n)), (1-\lambda, \delta), (a_j, A_j)_{n+1,r}\\ ((b_m, B_m)), (1-\lambda, \delta), b_j, B_j)_{m+1,s}\end{matrix}\right] \quad \ldots(2.1)$$

where $0 < \sigma \leqslant 1$, $\lambda > 0$.

Here and throughout this paper, we use the shorthand notations:

$$\nu = \frac{\lambda}{\delta},\ \theta = \frac{\alpha_p}{\beta_p},\ \varphi = \frac{\alpha_p + 1}{\beta_p + 1},\ \psi = \prod_{j=1}^{p} \frac{\Gamma(\beta_j)}{\Gamma(\alpha_j)} \quad \ldots(2.2)$$

and

$$M_1 = \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j\nu) \prod_{j=1}^{n} \Gamma(1 - a_j - A_j\nu)}{\prod_{j=m+1}^{s} \Gamma(1 - b_j - B_j\nu) \prod_{j=n+1}^{r} \Gamma(a_j + A_j\nu)} \quad \ldots(2.3)$$

and M_2 represents the expression M_1, where ν has been replaced by $(\nu + (1/\delta))$.

$$\frac{\theta^{-\lambda}}{\Gamma(\sigma)} H^{m+1,n+1}_{r+1,s+1}\left[z\theta^{-\delta}\,\middle|\,\begin{matrix}((a_n, A_n)), (1-\lambda, \delta), (a_j, A_j)_{n+1,r}\\ ((b_m, B_m)), (\sigma-\lambda, \delta), (b_j, B_j)_{m+1,s}\end{matrix}\right]$$

$$< \frac{\psi}{\Gamma(\sigma)} H^{m+p+1,n+1}_{r+p+1,s+p+1}\left[z\,\middle|\,\begin{matrix}((a_n, A_n)), (1-\lambda, \delta), ((\beta_p-\lambda, \delta)), (a_j, A_j)_{n+1,r}\\ ((b_m, B_m)), (\sigma-\lambda, \delta), ((\alpha_p-\lambda, \delta)), (b_j, B_j)_{m+1,s}\end{matrix}\right]$$

$$< (1-\theta) M_1 \delta^{-1} z^{-\nu}$$

$$+ \frac{\theta}{\Gamma(\sigma)} H^{m+1,n+1}_{r+1,s+1}\left[z\,\middle|\,\begin{matrix}((a_n, A_n)), (1-\lambda, \delta), (a_j, A_j)_{n+1,r}\\ ((b_m, B_m)), (\sigma-\lambda, \delta), (b_j\ B_j)_{m+1,s}\end{matrix}\right] \quad \ldots(2.4)$$

where $\sigma > 0$, $\lambda > 0$.

$$M_1\delta^{-1}z^{-\nu} - \sigma\theta\left(1-\frac{\varphi}{2}\right)M_2\delta^{-1}z^{-(\nu+(1/\delta))}$$

$$- \frac{\sigma\theta\varphi}{2\Gamma(\sigma+1)} H^{m+1,n+1}_{r+1,s+1}\left[z\,\middle|\,\begin{matrix}((a_n, A_n)), (-\lambda, \delta), (a_j, A_j)_{n+1,r}\\ ((b_m, B_m)), (\sigma-\lambda, \delta), (b_j, B_j)_{m+1,s}\end{matrix}\right]$$

$$< \frac{\psi}{\Gamma(\sigma)} H^{m+p+1,n+1}_{r+p+1,s+p+1}\left[z\,\middle|\,\begin{matrix}((a_n, A_n)), (1-\lambda, \delta), ((\beta_p-\lambda, \delta)), (a_j, A_j)_{n+1,r}\\ ((b_m, B_m)), (\sigma-\lambda, \delta), ((\alpha_p-\lambda, \delta)), (b_j, B_j)_{m+1,s}\end{matrix}\right]$$

$$< M_1\delta^{-1}z^{-\nu} - \frac{\sigma\theta}{\Gamma(\sigma+1)}\left(\frac{\varphi}{2}\right)^{-(\lambda+1)}$$

$$\times H^{m+1,n+1}_{r+1,s+1}\left[z\left(\frac{\varphi}{2}\right)^{-\delta}\,\middle|\,\begin{matrix}((a_n, A_n)), (-\lambda, \delta), (a_j, A_j)_{n+1,r}\\ ((b_m, B_m)), (\sigma-\lambda, \delta), (b_j, B_j)_{m+1,s}\end{matrix}\right] \quad ...(2.5)$$

where $\sigma > 0, \lambda > 0$.

$$\left(-\frac{1}{\sigma}\right)M_1\delta^{-1}z^{-\nu} + \left\{\left(\frac{\sigma+1}{\sigma}\right)\times\left(\frac{\theta}{\sigma+1}\right)^{-\lambda}\right\}$$

$$\times H^{m+1,n+1}_{r+1,s+1}\left[z\left(\frac{\theta}{\sigma+1}\right)^{-\delta}\,\middle|\,\begin{matrix}((a_n, A_n)), (1-\lambda, \delta), (a_j, A_j)_{n+1,r}\\ ((b_m, B_m)), (1-\lambda, \delta), (b_j, B_j)_{m+1,s}\end{matrix}\right]$$

$$< \Gamma(\sigma)(\psi) H^{m+p+1,n+1}_{r+p+2,s+p+1}\left[z\,\middle|\,\begin{matrix}((a_n, A_n)), (1-\lambda, \delta), (\sigma-\lambda, \delta), ((\beta_p-\lambda, \delta)), (a_j, A_j)_{n+1,r}\\ ((b_m, B_m)), (1-\lambda, \delta), ((\alpha_p-\lambda, \delta)), (b_j, B_j)_{m+1,s}\end{matrix}\right]$$

$$< \left(1-\frac{\theta}{\varphi}\right)M_1\delta^{-1}z^{-\nu} + \frac{\theta}{\varphi}\left(\frac{\varphi}{\sigma}\right)^{-\lambda}$$

$$\times H^{m+1,n+1}_{r+1,s+1}\left[z\left(\frac{\varphi}{\sigma}\right)^{-\delta}\,\middle|\,\begin{matrix}((a_n, A_n)), (1-\lambda, \delta), (a_j, A_j)_{n+1,r}\\ ((b_m, B_m)), (1-\lambda, \delta), (b_j, B_j)_{m+1,s}\end{matrix}\right] \quad ...(2.6)$$

where $\sigma > 0, \lambda > 0$.

$$\theta^{-\lambda}H^{m,n+1}_{r+1,s}\left[z\theta^{-\delta}\,\middle|\,\begin{matrix}((a_n, A_n)), (1-\lambda, \delta), (a_j, A_j)_{n+1,r}\\ ((b_s, B_s))\end{matrix}\right]$$

$$< \psi H^{m+p,n+1}_{r+p+1,s+p}\left[z\,\middle|\,\begin{matrix}((a_n, A_n)), (1-\lambda, \delta), ((\beta_p-\lambda, \delta)), (a_j, A_j)_{n+1,r}\\ ((b_m, B_m)), ((\alpha_p-\lambda, \delta)), (b_j, B_j)_{m+1,s}\end{matrix}\right]$$

$$< (1-\theta)M_1\delta^{-1}z^{-\nu}$$

$$+ \theta H^{m,n+1}_{r+1,s}\left[z\,\middle|\,\begin{matrix}((a_n, A_n)), (1-\lambda, \delta), (a_j, A_j)_{n+1,r}\\ ((b_s, B_s))\end{matrix}\right] \quad ...(2.7)$$

where $\lambda > 0$.

3. Proof

To prove (2.1) we replace z by t in (4.10) of Luke (1972, p. 53), multiply throughout by $t^{\lambda-1}H^{m,n}_{r,s}\left[zt^{\delta}\,\middle|\,\begin{matrix}((a_r, A_r))\\((b_s, B_s))\end{matrix}\right]$ and integrate with respect to t between 0 and ∞.

Now evaluating the integrals involved with the help of (5.1) of Gupta and Jain (1966, p. 601), we get the result.

Proceeding on similar lines the inequalities (2.4), (2.5), (2.6) and (2.7) can also be established by using the results of Luke [1972, pp. 55–57, (4.20), (4.22), (5.1) and (5.5)].

4. Particular Cases

Due to general character of inequalities discussed in this paper, new as well as known relations may be derived. Here we discuss the results recently given by Raina and Koul (1976).

(i) If we take $m = s = 1, n = r = 0, b_1 = 0, B_1 = 1$ and replace δ by ρ in the results (2.1), (2.4), (2.5), (2.6), (2.7) we get the relations due to Raina and Koul (1976, pp. 35–36, (2.1) to (2.5)].

(ii) On putting $m = s = n = r = 1, b_1 = 0, B_1 = 1, a_1 = 1 - \mu, A_1 = 1, \delta = \rho$ in (2.1), (2.4), (2.5), (2.6), (2.7) inequalities (2.6) to (2.10) due to Raina and Koul (1976, pp. 36–38) can be obtained.

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