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ON THE STRONG MATRIX SUMMABILITY OF ULTRASPHERICAL SERIES

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In the present paper, the author establishes a result for strong matrix summability of ultraspherical series which may be considered as an anologue of a recent theorem of Yadav (1979).

The summability of ultraspherical series by various ordinary and absolute summability methods has been discussed by a number of researchers like Kogbetliantz (1924), Gupta (1958), etc. But nothing seems to have been done so far in the direction of study of ultraspherical series by strong summability methods. In an attempt to make an advance study in this direction we, in the present paper, propose to establish a result for the strong matrix summability of ultraspherical series which may be considered as an anologue of a recent theorem on absolute Cesàro summability of ultraspherical series of Yadav (1979).

§1. The triangular matrix $(\Lambda) = (\lambda_{n,k})$, where $n = 0, 1, 2, 3, \ldots$ and $k = 0, 1, 2, 3, \ldots$ and $\lambda_{n,k} = 0$ for $k > n$, is regular (in the sense of defining a regular sequence to sequence transformation) if

$$\lim_{n\to\infty} \lambda_{n,k} = 0, \text{ for every fixed } k; \qquad \ldots(1.1)$$

$$\sum_{k=0}^{n} | \lambda_{n,k} | \leqslant M, \text{ independent of } n; \qquad \ldots(1.2)$$

and

$$\lim_{n\to\infty} \sum_{k} \lambda_{n,k} = 1. \qquad \ldots(1.3)$$

A series $\sum_{n=0}^{\infty} a_n$, with the sequence of partial sums $\{S_n\}$ is said to be strongly summable (Λ) or summable $[\Lambda]$ to the sum S, if

$$\sum_{k=0}^{n} \lambda_{n,k} | S_k - S | = o(1), \text{ as } n \to \infty.$$

In the following three cases :

(a) $\lambda_{n,k} = \dfrac{1}{n+1} \quad (k \leqslant n)$

(b) $$\lambda_{n,k} = \frac{1}{(k+1)\sum_{j=0}^{n}\frac{1}{j+1}} \quad (k \leqslant n)$$

(c) $$\lambda_{n,k} = \frac{1}{(n-k+1)\sum_{j=0}^{n}\frac{1}{j+1}} \quad (k \leqslant n)$$

summability $[\Lambda]$ becomes respectively Cesàro summability $[C, 1]$, a Riesz summability equivalent to $[R, \log n, 1]$ and Nörlund summability $\left[N, \frac{1}{n+1}\right]$.

The series Σa_n is said to be strongly summable (Λ) with index $q\ (q > 0)$, to the sum S if the sequence of partial sums $\{S_n\}$ is such that

$$\sum_{k=0}^{n} \lambda_{n,k} \mid S_k - S \mid^{q} = o(1), \text{ as } n \to \infty.$$

§2. Let $f(\theta, \phi)$ be a function defined for the range $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \phi \leqslant 2\pi$. The ultraspherical series corresponding to $f(\theta, \phi)$ on the surface S is

$$f(\theta, \phi) \sim \frac{1}{2\pi} \sum_{n=0}^{\infty} (n+\lambda) \iint_S \frac{f(\theta', \phi')\, P_n^{(\lambda)}(\cos\omega) \sin\theta'\, d\theta'\, d\phi'}{[\sin^2\theta' \sin^2(\phi - \phi')]^{(1-2\lambda)/2}} \quad \ldots(2.1)$$

where $\cos\omega = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$ and ultraspherical polynomials $P_n^{(\lambda)}(x)$ are defined by the following:

$$(1 - 2xt + t^2)^{-\lambda} = \sum_{n=0}^{\infty} t^n P_n^{(\lambda)}(x), \ \lambda > 0.$$

A generalized mean value of $f(\theta, \phi)$ on the sphere has been defined by Kogbetliantz (1924) as follows:

$$f(\omega) = \frac{1}{2\pi(\sin\omega)^{2\lambda}} \int_{C_w} \frac{f(\theta', \phi')\, ds'}{[\sin^2\theta' \sin^2(\phi - \phi')]^{(1-2\lambda)/2}} \quad \ldots(2.2)$$

where the integral is taken along the small circle C whose centre is (θ, ϕ) on the sphere S and whose curvilinear radius is ω.

The series (2.1) now reduces to

$$f(\theta, \phi) \sim \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\lambda)} \sum_{n=0}^{\infty} (n+\lambda) \int_0^{\pi} f(\omega) \sin^{2\lambda}\omega\, P_n^{(\lambda)}(\cos\omega)\, d\omega. \quad \ldots(2.3)$$

We write

$\phi(\omega) = (f(\omega) - A)(\sin\omega)^{2\lambda-1}$, where A is a fixed constant.

§3. We prove the following theorem:

Theorem — Let $\phi(\omega) \in BV(\eta, \pi)$ where $\eta = \frac{\mu}{n^{\Delta}}, \frac{1-\lambda}{\lambda} > \Delta > 0,\ 0 < \lambda < 1$ and μ is a large constant. If

$$\Phi(t) \equiv \int_0^t |\phi(\omega)|\, d\omega = O\left(\frac{t^{\alpha}}{(\log(1/t))^{1+\epsilon}}\right)_{\epsilon>0} \qquad ...(3.1)$$

where $\alpha = \dfrac{2\lambda + 1 - \Delta}{\Delta}$, as $t \to 0$,

then the series (2.1) is summable $[\Lambda]$ with index q ($q > 0$) to the value A, provided that (Λ) is regular.

§4. For the proof of the theorem we require following lemmas :

Lemma 1 (Szegö 1967, p. 171) — We have, for $\lambda > 0$

$$P_n^{(\lambda)}(\cos\theta) = \begin{cases} \theta^{-\lambda} O(n^{\lambda-1}), & c/n \leqslant \theta \leqslant \pi/2 \\ O(n^{2\lambda-1}), & 0 \leqslant \theta \leqslant c/n. \end{cases}$$

and

$$(\sin\theta)^{\lambda}\, |P_n^{(\lambda)}(\cos\theta)| < 2^{1-\lambda}\{\Gamma(\lambda)\}^{-1} n^{\lambda-1}, \quad \begin{matrix} 0 < \lambda < 1 \\ 0 \leqslant \theta \leqslant \pi \end{matrix}$$

Lemma 2 (Szegö 1967, p. 84) — For $n \geqslant 0$, we have

$$\frac{d}{dx}\{P_n^{(\lambda)}(x)\} = 2\lambda P_n^{(\lambda+1)}(x),\ P_{-1}^{(\lambda)}(x) = 0.$$

§5. *Proof of the Theorem* — Let S_n denote the nth partial sum of the series (2.1). Then we have (Szegö 1967, p. 84)

$$S_n = \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\lambda)} \int_0^{\pi} f(\omega) \sum_{k=0}^{n} (k+\lambda) P_k^{(\lambda)}(\cos\omega)(\sin\omega)^{2\lambda}\, d\omega$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\lambda)} \int_0^{\pi} f(\omega) \left[\frac{d}{dx}\{P_{n+1}^{(\lambda)}(x) + P_n^{(\lambda)}(x)\}\right]_{x=\cos\omega} \times (\sin\omega)^{2\lambda}\, d\omega.$$

Therefore

$$S_n - A = \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\lambda)} \int_0^{\pi} \phi(\omega) \frac{d}{d\omega} \{P_{n+1}^{(\lambda)}(\cos\omega) + P_n^{(\lambda)}(\cos\omega)\}\, d\omega$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\lambda)} \int_0^{\pi} \phi(\omega) \frac{d}{d\omega} P_{n+1}^{(\lambda)}(\cos\omega)\, d\omega$$

$$+ \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\lambda)} \int_0^{\pi} \phi(\omega) \frac{d}{d\omega} P_n^{(\lambda)}(\cos\omega)\, d\omega$$

$$= I_1 + I_2, \text{ say.}$$

For proving our theorem we have to show that

$$\sum_{\nu=1}^{n} \lambda_{n,\nu} \mid S_\nu - A \mid^q = o(1), \text{ as } n \to \infty.$$

Now applying Minkowski's inequality we get

$$\left\{ \sum_{\nu=1}^{n} \lambda_{n,\nu} \mid S_\nu - A \mid^q \right\}^{1/q} \leqslant \left\{ \sum_{\nu=1}^{n} \lambda_{n,\nu} \mid I_1 \mid^q \right\}^{1/q}$$

$$+ \left\{ \sum_{\nu=1}^{n} \lambda_{n,\nu} \mid I_2 \mid^q \right\}^{1/q}$$

$$= (\Sigma_1)^{1/q} + (\Sigma_2)^{1/q}, \text{ (say).}$$

Let us consider

$$I_1 = \left| \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\lambda)} \int_0^{\pi} \phi(\omega) \frac{d}{d\omega} P_{n+1}^{(\lambda)}(\cos\omega)\, d\omega \right|$$

$$= \left| \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\lambda)} \int_0^{\eta} \phi(\omega) \frac{d}{d\omega} P_{n+1}^{(\lambda)}(\cos\omega)\, d\omega \right|$$

$$+ \left| \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}+\lambda)} \int_\eta^{\pi} \phi(\omega) \frac{d}{d\omega} P_{n+1}^{(\lambda)}(\cos\omega)\, d\omega \right|$$

$$= \mid I_{1\cdot 1} \mid + \mid I_{1\cdot 2} \mid, \text{ say.}$$

But

$$| I_{1\cdot 1} | = O(n^{2\lambda+1}) \int_0^{\eta} \omega \, | \phi(\omega) | \, d\omega$$

$$= O(n^{2\lambda+1}) \, \eta \, . \, \eta^{\alpha} \frac{1}{(\log n)^{\epsilon+1}}$$

$$= O(n^{2\lambda+1}) \frac{n^{-\Delta(\alpha+1)}}{(\log n)^{\epsilon+1}}$$

$$= O(1/(\log n)^{1+\epsilon}), \quad \epsilon > 0$$

$$= o(1), \text{ as } n \to \infty.$$

And

$$| I_{1\cdot 2} | = \left| \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2}) \, \Gamma(\frac{1}{2} + \lambda)} \int_{\eta}^{\pi} \phi(\omega) \frac{d}{d\omega} \{P_{n+1}^{(\lambda)} (\cos \omega)\} \, d\omega \right|$$

$$= \left| \frac{\Gamma(\lambda)}{\Gamma(\frac{1}{2}) \, \Gamma(\frac{1}{2} + \lambda)} \left[- \int_{\eta}^{\pi} d\phi(\omega) \, \{P_{n+1}^{(\lambda)} (\cos \omega)\} + \{\phi(\omega) \, P_{n+1}^{(\lambda)} (\cos \omega)\}_{\eta}^{\pi} \right] \right|$$

$$= o(n^{\lambda-1}) \, \eta^{-\lambda}$$

$$= O(n^{\lambda-1+\Delta\lambda}) = O\left(\frac{1}{n^{1-\lambda-\Delta\lambda}}\right)$$

$$= o(1), \text{ as } n \to \infty.$$

Hence

$$| I_1 | = o(1), \text{ as } n \to \infty.$$

Therefore

$$\Sigma_1 = \sum_{\nu=1}^{n} \lambda_{n,\nu} \, | I_1 |^q$$

$$= o\left\{ \sum_{\nu=1}^{n} \lambda_{n,\nu} \right\}$$

$$= O(1), \text{ by (1.3), as } n \to \infty$$

and

$$(\Sigma_1)^{1/q} = o(1).$$

Following on the same lines as for $(\Sigma_1)^{1/q}$, we can show that

$$(\Sigma_2)^{1/q} = o(1).$$

This completes the proof of the theorem.

Remark : If $\alpha > \frac{2\lambda + 1 - \Delta}{\Delta}$, we can replace (3.1) by the following condition

$$\Phi(t) \equiv \int_0^t | \phi(\omega) | \, d\omega = O(t^{\alpha}).$$

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