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DUAL INTEGRAL EQUATIONS INVOLVING THE GENERALIZED FOX FUNCTION AS THE KERNEL

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Fractional integral operators are used to obtain a formal solution of dual integral equations having a special case of the multivariable H-function of Srivastava and Panda (1976) as kernel by reducing them to the ones with a common kernel. The results of Parashar and Goyal (1973), Jain and Goyal (1976), Goyal and Aggarwala (1974) and many others, follow as particular cases of the present investigation.

1. FORMULATION OF PROBLEM

In this paper we propose to obtain the formal solution of the following dual integral equations

$$\int_0^\infty \cdots \int_0^\infty H^*(x_1u_1, \ldots, x_nu_n)\, f(u_1, \ldots, u_n)\, du_1 \ldots du_n$$

$$= \phi'(x_1, \ldots, x_n);\ 0 < x_1, \ldots, x_n < 1 \qquad \ldots(1.1)$$

$$\int_0^\infty \cdots \int_0^\infty H_1^*\,(x_1u_1, \ldots, x_nu_n)\, f(u_1, \ldots, u_n)\, du_1 \ldots du_n$$

$$= \psi'(x_1, \ldots, x_n);\ x_1, \ldots, x_n > 1 \qquad \ldots(1.2)$$

where $\phi'(x_1, \ldots, x_n)$, $\psi'(x_1, \ldots, x_n)$ are given and $f(x_1, \ldots, x_n)$ is required to be determined. Also we assume that $H_1^*\,(u_1x_1, \ldots, u_nx_n)$ of (1.2) is of the same type as $H^*\,(u_1x_1, \ldots, u_nx_n)$ with c_j^k replaced by e_j^k for $j = 1, \ldots, p_k$; $\forall\ k \in (1, \ldots, n)$ and d_j^k replaced by f_j^k for $j = 1, \ldots, q_k$; $\forall\ k \in (1, \ldots, n)$. We also assume that H_1^* satisfies all the conditions given for H^* and have common contours with it.

The formal solution so obtained is

$$f(x_1, \ldots, x_n) = \int_0^\infty \cdots \int_0^\infty H^{**}(x_1u_1, \ldots, x_nu_n)\, t(u_1, \ldots, u_n)\, du_1, \ldots, du_n \qquad \ldots(1.3)$$

where

$$t(x_1, \ldots, x_n) = T^{*k}_{l_k+1}\,[T^{*k}_{l_k+2} \ldots T^{*k}_{q_k}\, T^k_1 \ldots T^k_{m_k}\,\{\phi'(x_1, \ldots, x_n)\}] \qquad 0 < x_1, \ldots, x_n < 1 \qquad \ldots(1.4)$$

$$= R^{*k}_{m_k+1}\,[R^{*k}_{m_k+2} \ldots R^{*}_{p_k}\, R^k_1 \ldots R^k_{l_k}\,\{\psi'(x_1, \ldots, x_n)\}] \qquad x_1, \ldots, x_n > 1 \qquad \ldots(1.5)$$

also

$$H^{**}(x_1, \ldots, x_n) = H^{p-m,0;p_1-m_1,q_1-l_1;\cdots;p_n-m_n,q_n-l_n}_{p,0;p_1,q_1;\cdots;p_n,q_n}$$

$$\left[\begin{array}{c|l} x_1 & [(1 - a_{m+1,p} - \sum_{k=1}^{n} \alpha^{t_k}_{m+1,p};\ \alpha^{t_1}_{m+1,p}, \ldots, \alpha^{t_n}_{m+1,p})] \\ \vdots & \{(1 - c^1_{m_1+1,p_1} + \gamma^1_{m_1+1,p_1}, \gamma^1_{m_1+1,p_1}), (1 - e^1_{m_1} + \gamma^1_{m_1}, \gamma^1_{m_1})\}; \ldots; \\ x_n & \{(1 - c^n_{m_n+1,p_n} + \gamma^n_{m_n+1,p_n}, \gamma^n_{m_n+1,p_n}), (1 - e^n_{m_n} + \gamma^n_{m_n}, \gamma^n_{m_n})\} : \\ & \{(1 - f^1_{l_1+1,q_1} + \delta^1_{l_1+1,q_1}, \delta^1_{l_1+1,q_1}), (1 - d^1_{l_1} + \delta^1_{l_1}, \delta^1_{l_1})\}; \ldots; \\ & \{(1 - f^n_{l_n+1,q_n} + \delta^n_{l_n+1,q_n}, \delta^n_{l_n+1,q_n})\,(1 - d^n_{l_n} + \delta^n_{l_n}, \delta^n_{l_n})\} : \end{array}\right] \qquad \ldots(1.6)$$

It is clear from the nature of the solution that it can be written by inspection from eqns. (1.1) and (1.2)

$$f(x_1, \ldots, x_n) = \int_0^1 \ldots \int_0^1 H^{**}(u_1x_1, \ldots, u_nx_n)\, T^{*k}_{l_k+1}\,[T^{*k}_{l_k+2} \ldots T^{*k}_{q_k}\, T^k_1 \ldots$$

$$\ldots T^k_{m_k}\,\{\phi'(u_1, \ldots, u_n)\}]\, du_1 \ldots du_n$$

$$+ \int_1^\infty \ldots \int_1^\infty H^{**}(x_1u_1, \ldots, x_nu_n)\, R^{*k}_{m_k+1}\,[R^{*k}_{m_k+2} \ldots R^{*k}_{p_k}\, R^k_1 \ldots$$

$$\ldots R^k_{l_k}\,\{\psi'(u_1, \ldots, u_n)\}]\, du_1 \ldots du_n. \qquad \ldots(1.7)$$

Here the $H^*(x_1u_1, \ldots, x_nu_n)$ is a particular case of the multivariable H-function of Srivastava and Panda [1976, eqns. (1.3) - (1.7), p. 130] in the contracted notation (slightly different)

$$H^{m,0;(m_1,l_1);\cdots;(m_n,l_n)}_{p,0;(p_1,q_1);\cdots;(p_n,q_n)}\left[\begin{array}{c|l} x_1 & [a_p;\ \alpha^{t_1}_1, \ldots, \alpha^{t_n}_n] : \\ \vdots & \{(c^1_{p_1}, \gamma^1_{p_1})\}; \ldots; \{(c^n_{p_n}, \gamma^n_{p_n})\} : \\ x_n & \{(d^1_{q_1}, \delta^1_{q_1})\}; \ldots; \{(d^n_{q_n}, \delta^n_{q_n})\} : \end{array}\right] =$$

(equation continued on p. 1178)

$$= (1/2\pi i)^n \int_{L_1} \dots \int_{L_n} \psi(\sum_{k=1}^{n} s_k) \prod_{k=1}^{n} \{\phi_k(s_k)\, x_k^{-s_k}\, ds_k\} \qquad \dots(1.8)$$

where

$$\phi_k(s_k) = \prod_{j=1}^{m_k} \Gamma(1 - c_j^k + \gamma_j^k s_k) \prod_{j=1}^{l_k} \Gamma(d_j^k - \delta_j^k s_k)$$

$$\times \Big[\prod_{j=1+m_k}^{p_k} \Gamma(c_j^k - \gamma_j^k s_k) \prod_{j=1+l_k}^{q_k} \Gamma(1 - d_j^k + \delta_j^k s_k)\Big]^{-1} \qquad \dots(1.9)$$

$$\psi(\sum_{k=1}^{n} s_k) = \prod_{j=1}^{m} \Gamma(a_j + \sum_{k=1}^{n} \alpha_j^{ik} s_k) \Big[\prod_{j=1+m}^{p} \Gamma(1 - a_j - \sum_{k=1}^{n} \alpha_j^{ik} s_k)\Big]^{-1}. \qquad \dots(1.10)$$

The integers m, p, (m_k), (l_k), (p_k), (q_k) $\forall\, k \in (1, \dots, n)$ satisfy the inequalities $0 \leqslant m \leqslant p$; $0 \leqslant m_k \leqslant p_k$; $0 \leqslant l_k \leqslant q_k$; $q_k \geqslant 0$, $\forall\, k \in (1, \dots, n)$. The values $x_k = 0$, $\forall\, k \in (1, \dots, n)$ are excluded. An empty product is interpreted as unity. The contour L_k is in the s_k plane for every $k \in (1, \dots, n)$ and runs from $-i\infty$ to $+i\infty$ with loops if necessary, to ensure that the poles of $\Gamma(a_j + \sum_{k=1}^{n} \alpha_j^{ik} s_k)$ for $j = 1, \dots, m$ and $\Gamma(1 - c_j^k + \gamma_j^k s_k)$ for $j = 1, \dots, m_k$ lie to the left of the contour L_k and that of $\Gamma(d_j^k - \delta_j^k s_k)$ for $j = 1, \dots, l_k$ lie to the right of the contour L_k. Also $|\arg x_k| < \lambda^* \pi$, $\lambda^* > 0$ where

$$\lambda^* = \sum_{j=1}^{l_k} |\delta_j^k| + \sum_{j=1+m}^{p} |\alpha_j^{ik}| - \sum_{j=1+m_k}^{p_k} |\gamma_j^k|$$

$$+ \tfrac{1}{2}\Big[\sum_{j=1}^{p} \alpha_j^{ik} + \sum_{j=1}^{p_k} \gamma_j^k + \sum_{j=1}^{q_k} \delta_j^k \Big]$$

(α_p^{ik}), $(\gamma_{p_k}^k)$, $(\delta_{q_k}^k)$, $\forall\, k \in (1, \dots, n)$ are all positive.

Using the multidimensional Mellin transform of $H(x_1, \dots, x_n)$ given by Srivastava and Panda (1978, Lemmas 1 and 2, p. 125), constructing the Parseval theorem for n variables, viz.

If $\quad M\{y(u_1, \dots, u_n)\} = Y(s_1, \dots, s_n)$

and

$$M\{f(x_1 u_1, \dots, x_n u_n)\} = \prod_{k=1}^{n} (x_k^{-s_k})\, F(s_1, \dots, s_n)$$

then

$$\int_0^\infty \cdots \int_0^\infty y(x_1u_1, \ldots, x_nu_n)\, f(u_1, \ldots, u_n)\, du_1 \ldots du_n$$

$$= (1/2\pi i)^n \int_{-i\infty}^{+i\infty} \cdots \int_{-i\infty}^{+i\infty} Y(s_1, \ldots, s_n)$$

$$\times F(1 - s_1, \ldots, 1 - s_n) \prod_{k=1}^{n} (x_k^{-s_k}\, ds_k). \qquad \ldots(1.11)$$

On applying (1.11) to (1.1) and (1.2) we obtain

$$(1/2\pi i)^n \int_{L_1} \cdots \int_{L_n} \psi(\sum_{k=1}^{n} s_k)\, F(1 - s_1, \ldots, 1 - s_1) \prod_{k=1}^{n} \{\phi_k(s_k)\, x_k^{-s_k}\, ds_k\}$$

$$= \phi'(x_1, \ldots, x_n);\ \text{where}\ 0 < x_k < 1,\ \forall\, k \in (1, \ldots, n) \qquad \ldots(1.12)$$

and

$$(1/2\pi i)^n \int_{L_1} \cdots \int_{L_n} \psi(\sum_{k=1}^{n} s_k)\, F(1 - s_1, \ldots, 1 - s_n) \prod_{k=1}^{n} \{\phi_k^*(s_k)\, x_k^{-s_k}\, ds_k\}$$

$$= \psi'(x_1, \ldots, x_n);\ x_k > 1,\ \forall\, k \in (1, \ldots, n) \qquad \ldots(1.13)$$

and $\phi_k^*(s_k)$ represents $\phi_k(s_k)$ with c_j^k replaced by e_j^k for $j = 1, \ldots, p_k$; $\forall\, k \in (1, \ldots, n)$ and d_j^k replaced by f_j^k, $j = 1, \ldots, q_k$; $\forall\, k \in (1, \ldots, n)$.

2. Reduction of (1.12) and (1.13) into one with a Common Kernel

The integral operators are used to transform (1.12) and (1.13) into two others with a common kernel. Thus we transform

$$\prod_{k=1}^{n} \left[\frac{\prod_{j=1}^{m_k} \Gamma(1 - c_j^k + \gamma_j^k s_k)}{\prod_{j=1+l_k}^{q_k} \Gamma(1 - d_j^k + \delta_j^k s_k)} \right] \text{ of } (1.12)$$

into

$$\prod_{k=0}^{n} \left[\frac{\prod_{j=1}^{m_k} \Gamma(1 - e_j^k + \gamma_j^k s_k)}{\prod_{j=1+l_k}^{q_k} \Gamma(1 - f_j^k + \delta_j^k s_k)} \right] \text{ of } (1.13)$$

$$\prod_{k=1}^{n} \left[\frac{\prod_{j=1}^{l_k} \Gamma(f_j^k - \delta_j^k s_k)}{\prod_{j=1+m_k}^{p_k} \Gamma(e_j^k - \gamma_j^k s_k)} \right] \text{ of } (1.13)$$

into $$\prod_{k=1}^{n}\left[\frac{\prod_{j=1}^{l_k}\Gamma(d_j^k-\delta_j^k s_k)}{\prod_{j=1+m_k}^{p_k}\Gamma(c_j^k-\gamma_j^k s_k)}\right]$$ of (1.12).

In making these transformations we use the fractional integral operators T and R defined by Erdélyi (1950–51), viz.

$$T[\alpha,\beta;\gamma:w(x)]=(\gamma/\Gamma\alpha)\,x^{-\gamma\alpha+\gamma-\beta-1}\int_0^x(x^\gamma-v^\gamma)^{\alpha-1}\,v^\beta w(v)\,dv$$

$$R[\alpha,\beta;\gamma:w(x)]=(\gamma/\Gamma\alpha)\,x^\beta\int_0^\infty(v^\gamma-x^\gamma)^{\alpha-1}\,v^{-\beta-\gamma\alpha+\gamma-1}w(v)\,dv$$

provided $w(x)\in L_p(0,\infty)$, $p'\geqslant 1$; $\alpha>0$; $\beta>(1-p')/p'$. If in addition $w(x)$ can be differentiated sufficiently often, T and R exist for negative as well as positive (see also Fox 1965). In this case for brevity we write

$$T[e_j^k-c_j^k,\,(1-e_j^k)(-\gamma_j^k)^{-1}-1;\;(-\gamma_j^k)^{-1}:w(x)]=T_j^k[w(x)]$$

$$T[d_j^k-f_j^k,\,(1-d_j^k)(-\delta_j^k)^{-1}-1;(-\delta_j^k)^{-1}:w(x)]=T_j^{*k}[w(x)]$$

$$R[f_j^k-d_j^k,\,d_j^k(-\delta_j^k)^{-1};(-\delta_j^k)^{-1}:w(x)]=R_j^k[w(x)]$$

$$R[c_j^k-e_j^k,\,e_j^k(-\gamma_j^k)^{-1};(-\gamma_j^k)^{-1}:w(x)]=R_j^{*k}[w(x)].$$

Now, replace x_1 by v_1, multiply by $(v_1)^A(x_1^\theta-v_1^\theta)^{\theta_1}$ where $A=(1-e_{m_1}^1)\theta-1$, $\theta=(-\gamma_{m_1}^1)^{-1}$ and $\theta_1=e_{m_1}^1-c_{m_1}^1-1$, integrate with respect to v_1 under the limit $0<v_1<x_1$, $0<x_1<1$ and apply the well-known beta function formula to obtain

$$T_{m_1}^1[\phi'(x_1,x_2,\ldots,x)]$$

$$=\frac{x_1^Z}{\theta\Gamma(e_{m_1}^1-c_{m_1}^1)}\int_0^{x_1}(v_1)^A(x_1^\theta-v_1^\theta)^{\theta_1}\,\phi'(v_1,x_2,\ldots,x_n)\,dv_1$$

(here $Z=\theta c_{m_1}^1$).

On transforming it successively by the application of the operator T_j^k for $j=m_k-1$, $m_k-2,\ldots,3,2,1$ for $k=1$ and $j=m_k,m_k-1,\ldots,3,2,1$ for $k=2,3,\ldots,n$ and then applying the operator T_j^{*k}(successively) for $j=q_k,q_k-1,q_k-2,\ldots,l_k+3$,

$l_k+2,\ l_k+1;\ \forall\ k\in(1,\ldots,n)$ we finally get the right-hand side of $t(x_1,\ldots,x_n)$ given in (1.4).

Similarly for making the second transformation after replacement as in first transformation multiply by

$$(v_1)^B\,(v_1^{\theta^*}-x_1^{\theta^*})^{\theta_1^*} \text{ where } B=(1-f_{l_1}^1)\,\theta^*-1,\ \theta^*=(-\delta_{l_1}^1)^{-1}$$

and $$\theta_1^*=f_{l_1}^1-d_{l_1}^1-1.$$

Integrate with respect to v_1 from x_1 to ∞ ($x_1>1$) and apply the well-known beta function formula to obtain

$$R_{l_1}^1\,[\psi'(x_1,\ldots,x_n)]$$

$$=\frac{x_1^Y}{\theta^*\Gamma(f_{l_1}^1-d_{l_1}^1)}\int_{x_1}^{\infty}(v_1)^B\,(v_1^{\theta^*}-x_1^{\theta^*})^{\theta_1^*}\,\psi'(v_1,x_2,\ldots,x_n)\,dv_1$$

(Here $Y=\theta^* d_{l_1}^1$).

On transforming the above successively by the application of the operators R_j^k for $j=l_k-1,\ l_k-2,\ldots,2,1$ for $k=1$ and $j=l_k,\ l_k-1,\ldots,2,1$ for $k=2,3,4,\ldots,n$ and applying R_j^{*k} successively for $j=p_k,\ p_k-1,\ldots,m_k+2,\ m_k+1,\ \forall\ k\in(1,\ldots,n)$ we finally get $t(x_1,\ldots,x_n)$ given in (1.5). Thus the equations can be rewritten in the form

$$t(x_1,\ldots,x_n)=(1/2\pi i)^n\int_{(L_n)}\psi\Big(\sum_{k=1}^{n}s_k\Big)\,F(1-s_1,\ldots,1-s_n)$$

$$\prod_{k=1}^{n}\left[\frac{\prod_{j=1}^{m_k}\Gamma(1-e_j^k+\gamma_j^k s_k)\prod_{j=1}^{l_k}(d_j^k-\delta_j^k s_k)\,x_k^{-s_k}ds_k}{\prod_{j=1+m_k}^{p_k}\Gamma(c_j^k-\gamma_j^k s_k)\prod_{j=1+l_k}^{q_k}\Gamma(1-f_j^k+\delta_j^k s_k)}\right]$$

The above equation is the reduction of (1.12) and (1.13) into the one with a common kernel. On treating the kernel of the above equation as an unsymmetrical Fourier kernel, and following the procedure adopted by Fox (1965), we easily arrive at the solution given in (1.3).

3. Particular Cases

(i) If $\alpha_{4,n}^{t_{4,n}} = \beta_{4,n}^{t_{4,n}} = \gamma_{p_{4,n}}^{k} = \delta_{q_{4,n}}^{k} = 0, x_1 = x, x_2 = y, x_3 = z, p_{4,n} = q_{4,n} = 0$ and take the limit as $x_{4,n}$ tends to zero, we obtain the dual integral equations and the solutions given by Jain and Goyal (1976) which in turn contain the results of Parashar and Goyal (1973), Saxena (1967) as corollaries.

(ii) If α's, β's, γ's, δ's are all unity $\forall\, k \in (1, \ldots, n)$, the main result of Goyal and Aggarwala (1974) becomes a corollary of the present investigation.

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