Indian J. pure appl. Math., **11**(9) : 1183–1187, September 1980

ON COMPOSITION OF INTEGRAL OPERATORS WITH FOURIER TYPE KERNELS

V. M. BHISE AND MADHAVI DIGHE

Department of Mathematics, G. S. Institute of Technology and Science, Indore 452003, *M.P.*

(*Received* 22 *March* 1979; *after revision* 18 *December* 1979)

In this paper an integral operator with a Fourier type kernel is expressed as the composition of two operators, one of which is a fractional integral operator and the other an operator with a Fourier type kernel. The composition is also estimated as a bounded operator from $L^p(R^+)$ to $L^q(R^+)$.

§1. Composition of different operators has been studied by Okikiolu (1966) while Srivastava and Buschman (1973) have studied the composition of fractional integral operators. In this paper we study the composition of an operator with the Fourier type kernel and a fractional integral operator. Special cases have been obtained which lead to known results by Okikiolu (1966). In the next section, we define the operators, find their composition and give their special cases. In section 3 we establish a theorem for the estimation of the composition of the operators, by treating the composition as a Fourier type integral.

§2. An integral operator is defined by Okikiolu (1971, p. 334) as a Fourier type integral or an operator having a Fourier type kernel, on suitable spaces of measurable functions on R, if

$$Tf(x) = x^{\sigma} \int_R t^{v} k(xt)\, f(t)\, dt$$

where $k(t)$ is a suitable kernel. Okikiolu (1966) has obtained the composition of a fractional integral operator and Fourier type integral operators having the sine and cosine kernels. We have generalized these results by considering the H-function in the kernels of the Fourier type integral operators. We have the known integral (Anandani 1968)

$$H^{m,n+1}_{p+1,q+1}\left[c(x^2t^2)^k \,\middle|\, \begin{matrix} (1+\frac{1}{2}a-\frac{1}{2}k^{-1}, 1), \{(a_p+\frac{1}{2}aA_p, A_p)\} \\ \{(b_q+\frac{1}{2}a\,B_q, B_q)\}, (\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}k^{-1}, 1) \end{matrix}\right]$$

$$= \frac{2k}{\Gamma(a+\frac{1}{2})}\, c^{a/2} t^{ka} x^{-ka+k-1} \int_0^x (x^{2k}-y^{2k})^{(2a-1)/2}$$

$$\times\, H^{m,n}_{p,q}\left[c(t^2y^2)^k \,\middle|\, \begin{matrix} \{(a_p, A_p)\} \\ \{(b_q, B_q)\} \end{matrix}\right] dy \qquad \ldots(2.1)$$

where $\{(a_p, A_p)\}$ stands for the set of parameters $(a_1, A_1), (a_2, A_2), \ldots (a_p, A_p)$; $a + \frac{1}{2} > 0$; $c, k > 0$; m, n, p, q are integers satisfying $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$; A_j and B_j are all positive.

$$\phi = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \sum_{j=n+1}^{p} A_j - \sum_{j=m+1}^{q} B_j > 0, \ | \arg c(t^2y^2)^k | < \tfrac{1}{2}\phi\pi,$$

$$\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0,\ 1/2k > -\min \operatorname{Re}(b_h/B_h),\ 1 \leqslant h \leqslant m.$$

Multiplying eqn. (2.1) by $t^v f(t)$, integrating with respect to t and changing the order of integration in the right hand side, it follows from Fubini theorem that for $t^{v+ka}f(t)$ in $L^1(R^+)$,

$$\int_0^\infty t^v H^{m,n+1}_{p+1,q+1}\left[c(x^2t^2)^k \,\middle|\, \begin{matrix}(1+\frac{1}{2}a-\frac{1}{2}k^{-1}, 1)\,\{(a_p+\frac{1}{2}aA_p, A_p)\}\\ \{(b_q+\frac{1}{2}aB_q, B_q)\}, (\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}k^{-1}, 1)\end{matrix}\right] f(t)\, dt$$

$$= \frac{2kc^{a/2}}{\Gamma(a+\frac{1}{2})} x^{-ka+k-1} \int_0^x (x^{2k}-y^{2k})^{(2a-1)/2}\, dy \int_0^\infty t^{v+ka} f(t)$$

$$\times H^{m,n}_{p,q}\left[c(t^2y^2)^k \,\middle|\, \begin{matrix}\{(a_p, A_p)\}\\ \{(b_q, B_q)\}\end{matrix}\right] dt. \qquad \ldots(2.2)$$

Now, denoting the fractional integral operator R_k by

$$R_k f(x) = x^{v+\sigma-ka+k-1} \int_0^x (x^{2k}-y^{2k})^{(2a-1)/2} y^{-\sigma-v-ka} f(y)\, dy \qquad \ldots(2.3)$$

and defining the Fourier type integral operators involving the H-functions as

$$T_H f(x) = \frac{\Gamma(a+\frac{1}{2})}{2kc^{a/2}} x^{v+\sigma} \int_0^\infty t^v f(t)$$

$$\times H^{m,n+1}_{p+1,q+1}\left[c(x^2t^2)^k \,\middle|\, \begin{matrix}(1+\frac{1}{2}a-\frac{1}{2}k^{-1}, 1), \{(a_p+\frac{1}{2}aA_p, A_p)\}\\ \{(b_q+\frac{1}{2}aB_q, B_q)\}, (\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}k^{-1}, 1)\end{matrix}\right] dt \qquad \ldots(2.4)$$

$$H^v_\sigma f(x) = x^{v+\sigma} \int_0^\infty t^v f(t)\, H^{m,n}_{p,q}\left[c(x^2t^2)^k \,\middle|\, \begin{matrix}\{(a_p, A_p)\}\\ \{(b_q, B_q)\}\end{matrix}\right] dt \qquad \ldots(2.5)$$

we have from (2.2)

$$T_H f(x) = R_k H^{v+ka}_\sigma f(x). \qquad \ldots(2.6)$$

Thus the operator T_H can be expressed as the composition of the operators R_k and H_σ^{v+ka}.

Special cases

(i) If all the A_j and B_j in (2.4) and (2.5) are equal to 1, we get from (2.6) the composition of operators involving the G-functions.

(ii) Further in (i), $k = 1$, $c = \frac{1}{4}$, $m = 1$, $n = 0$, $p = 0$, $q = 2$, $b_1 = 0$, $b_2 = \frac{1}{2}$, we get

$$T_J f(x) = RC_\sigma^{v+a} f(x). \qquad \text{...(2.7)}$$

which is the result given by Okikiolu (1966).

(iii) Again, by setting in (i) $k = 1$, $c = \frac{1}{4}$, $m = 1$, $n = 0$, $p = 0$, $q = 2$, $b_1 = \frac{1}{2}$, $b_2 = 0$, we get

$$T_H f(x) = RS_\sigma^{v+a} f(x) \qquad \text{...(2.8)}$$

which is another known result given Okikiolu (1966).

§3. In this section we establish the following theorem for the estimation of the Fourier type integral operator T_H.

Theorem — Let $f \in L^p(R^+)$, $p \geqslant 1$, and let $q \geqslant 1$, $1/q = 1 - (1/p) - \sigma > 0$, $0 < \sigma + (2/p) - 1 \leqslant 1/p$, then $T_H f(x)$ given by (2.4) can be extended as a bounded operator from $L^p(R^+)$ to $L^q(R^+)$ and $\| T_H \|_{p \to q} \leqslant \| T_H V_p \|_{p \to q}$, where

$$V_p f(x) = x^{-2/p} f(x^{-1}) \text{ if } a + \tfrac{1}{2} > 0 \qquad \text{...(3.1)}$$

and

$$-2k \min_{1 \leqslant h \leqslant m} \operatorname{Re}(b_h/B_h) < 1 + v + ka - (1/p)$$

$$< 2k\,[1 - \max_{1 \leqslant j \leqslant n} \operatorname{Re}\{a_j/A_j, (2k - 1)/2k\}]. \qquad \text{...(3.2)}$$

PROOF : The Fourier type integral operator (2.4) can be written as

$$T_H f(x) = x^\sigma \int_0^\infty g(xt)\, f(t)\, dt$$

where

$$g(xt) = \frac{\Gamma(a + \frac{1}{2})}{2kc^{a/2}} (xt)^v$$

$$\times H_{p+1, q+1}^{m, n+1} \left[c(x^2 t^2)^k \middle| \begin{matrix} (1 + \frac{1}{2}a - \frac{1}{2}k^{-1}, 1), \{(a_p + \frac{1}{2}aA_p, A_p)\} \\ \{(b_q + \frac{1}{2}aB_q, B_q)\}, (\frac{1}{2} - \frac{1}{2}a - \frac{1}{2}k^{-1}, 1) \end{matrix} \right]. \qquad \text{...(3.3)}$$

Using Theorem 4.2.9 of Okikiolu (1971, p. 197)

$$T_H V_p f(x) = x^{\sigma} \int_0^{\infty} t^{(2-2p)/p}\, g(x/t)\, f(t)\, dt$$

where V_p is given by (3.1).

Hence
$$T_H V_p f(x) = x^{\mu-1} \int_0^{\infty} g(x/t)\, f(t)\, dt$$

where

$$\mu = \sigma + 2p^{-1} - 1 \quad \text{and} \quad g(x/t) = (x/t)^{(2p-2)/p}\, g(x/t). \qquad \ldots(3.4)$$

Assuming $f \in L^p(R^+)$, $p \geqslant 1$, $q \geqslant 1$, $1/q = (1/p) - \mu = 1 - (1/p) - \sigma$, $0 \leqslant \mu \leqslant 1/p$, and using Theorem 4.2.4 of Okikiolu (1971, p. 197) we obtain

$$\| T_H V_p f \|_q \leqslant k_1^{1-\mu} \| f \|_p$$

where

$$k_1 = \int_0^{\infty} t^{(\mu-2+(1/p))/(1-\mu)}\, |\, g_1(t)\, |^{1/(1-\mu)}\, dt$$

$$= \left[\frac{\Gamma(a+\frac{1}{2})}{2kc^{a/2}}\right]^{1/(1-\mu)} \int_0^{\infty} t^{(\nu+\mu-(1/p))/(1-\mu)}$$

$$\times \left\{ H^{m,n+1}_{p+1,q+1}\left[ct^{2k} \,\middle|\, \begin{matrix} (1+\frac{1}{2}a-\frac{1}{2}k^{-1}, 1), \{(a_p+\frac{1}{2}aA_p, A_p)\} \\ \{(b_q+\frac{1}{2}aB_q, B_q)\}, (\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}k^{-1}, 1) \end{matrix} \right] \right\}^{1/(1-\mu)} dt$$

by using (3.3), (3.4). Also k_1 is finite if

$$-2k \min_{1 \leqslant h \leqslant m} \mathrm{Re}\,(b_h/B_h) < 1 + \nu + ka - (1/p)$$
$$< 2k\,[1 - \max_{1 \leqslant j \leqslant n} \mathrm{Re}\,\{a_j/A_j, (2k-1)/2k\}].$$

Hence $T_H V_p$ is a bounded operator from $L^p(R^+)$ to $L^q(R^+)$. Therefore by Theorem 4.2.9 of Okikiolu (1971, p. 197), T_H is also a bounded operator from $L^p(R^+)$ to $L^q(R^+)$ and $\| T_H \|_{p\to q} \leqslant \| T_H V_p \|_{p\to q}$, provided $T_H f$ and $T_H V_p f$ have a meaning.

Special Cases

In (3.2), setting the parameters as in (2.7) and (2.8), the operators T_J and T_H given by Okikiolu (1966) can be extended from $L^p(R^+)$ to $L^q(R^+)$ under the conditions $1/q = 1 - (1/p) - \sigma > 0$, $a + \frac{1}{2} > 0$, $1/p > \nu + a$.

ACKNOWLEDGEMENT

The authors are thankful to the referee for some useful suggestions.

References

Anandani, P. (1968). A study of the H-function and generalised Legendre's associated functions. Ph.D. Thesis; University of Indore, Indore.

Okikiolu, G. O. (1966). On integral operators with kernels involving Bessel functions. *Proc. Camb. phil. Soc.*, **62**, 477–84.

——— (1971). Aspects of the Theory of Bounded Integral Operators in L^p-Spaces. Academic Press, New York.

Srivastava, H. M., and Buschman, R. G. (1973). Composition of fractional integral operators involving Fox's H-function. *Acta Mexicana Ci. Tecn.*, **7**, 21–28.