*Indian J. pure appl. Math.*, **11**(9) : 1188–1193, September 1980

# CONDITIONS ON ρ-OID OPERATORS IMPLYING NORMALITY

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(*Received* 7 *June* 1978; *after revision* 19 *December* 1979)

Operators of the ρ-oid type are investigated in the context of similarity. The problem that under what conditions a ρ-oid operator can become a unitary operator is researched into. Analogous results involving essential similarity is also accomplished. A result of Furuta-Nakamoto is extended to ρ-oid operators. Some conditions on an operator that convert it into a normal operator are discussed. Two open problems are posed; sample : Is ρ-convexoid ($\rho > 2$) operator with real spectrum self-adjoint?

## 1. INTRODUCTION

Let $\beta(H)$ denote the Banach algebra of all operators on a Hilbert space $H$. An operator $T \in \beta(H)$ is said to be of class $C_\rho$, ($\rho > 0$) if there exists a unitary operator $U$ on some Hilbert space $K$ ($\supset H$) such that $T^n x = \rho\, PU^n x$, ($n = 1, 2, \ldots; x \in H$). Much information is to be found about this class in Sz.-Nagy and Foias (1970). Holbrook (1968) defined operator radius of an operator $T$ by

$$w_\rho(T) = \inf\{u : u > 0, u^{-1} T \in C_\rho\} \ (0 < \rho < \infty)$$

and

$$w_\infty(T) = r(T), \text{ the spectral radius of } T.$$

Holbrook (1968) characterized $C_\rho$ operators in terms of operator radii. Lin (1974) was led to introduce the notion of generalized numerical range $W_\rho(T)$ of an operator $T \in \beta(H)$ as

$$W_\rho(T) = \bigcap_{v \in \mathbb{C}} \{u \in \mathbb{C} : |u - v| \leqslant w_\rho(T - vI)\},\ 1 \leqslant \rho \leqslant \infty.$$

Phadke and Thakare (1978a, b) have also investigated generalized numerical ranges and related concepts.

The class of spectraloid operators on $H$ is known to be much larger. However, using the notion of $w_\rho(T)$ a much more wider class of operators, called ρ-oid operators, has been introduced.

*Definition* (Furuta 1969) — An operator $T \in \beta(H)$ is called ρ-oid if

$$w_\rho(T^k) = (w_\rho(T))^k, \; k = 1, 2, \ldots$$

[Clearly 1-oid and 2-oid operators are normaloid and spectraloid respectively.]

In section 2, our main concern is to obtain conditions on ρ-oid operators so that they become unitary. In the process, we have extended the results of DePrima (1974), and Khasbardar and Thakare (1978).

In section 3, we investigate conditions implying normality of an operator.

## 2. Conditions Implying Uniticity

In order to prove our first theorem, we need the following known results:

*Lemma* 1 (Stampfli 1969, Cor. 4) — If $T \in C_\alpha$, $T^{-1} \in C_\beta$ $(\alpha, \beta \geqslant 1)$, then $T$ is unitary.

*Lemma* 2 (Patel 1973, Th. 5) — Let $T$ be a left invertible operator with a left inverse $T_1$. If there exists an operator $S$ such that $T^* = S^{-1}T_1^p S$, $0 \notin \overline{W(S)}$ and, $p$ is a nonnegative integer, then the spectrum $\sigma(T)$ lies in the unit disc.

*Theorem* 1 — Let $T$ be an invertible operator such that (i) $T^* = S^{-1}T^{-p}S$ with $0 \notin \overline{W(S)}$, $p$ being some integer, and (ii) $T$ is ρ-oid, $T^{-1}$ is δ-oid, $(\rho, \delta \geqslant 1)$. Then $T$ is unitary.

Proof : An invertible operator is, in particular, left invertible. Hence, by Lemma 2, $\sigma(T)$ lies on the unit disc $E = \{z \in \mathbb{C} : |z| \leqslant 1\}$ of the complex plane. This implies that $r(T) \leqslant 1$. Since $T$ is ρ-oid $w_\rho(T) = r(T) \leqslant 1$; and $T \in C_\rho$.

As $\sigma(T) = \sigma_{ap}(T) \cup \sigma_{ap}(T^*)^*$, where $\sigma_{ap}(T)$ is the approximate point spectrum, and the star in $\mathbb{C}$ denotes conjugation, it follows that $\sigma_{ap}(T) \subset E$ and $\sigma_{ap}(T^*) \subset E$; and thus $\sigma(T^*) = \sigma_{ap}(T^*) \cup \sigma_{ap}(T)^* \subset E$.

In view of (i) and the fact that similarity preserves the spectra it is evident that $\sigma(T^{-p}) \subset E$ But at $\sigma(T^{-p}) = \sigma(T^{-1})^p$, it follows that $r(T^{-1}) \leqslant 1$. Again $T^{-1}$ is δ-oid and so $w_\delta(T^{-1}) = r(T^{-1}) \leqslant 1$ which implies that $T^{-1} \in C_\delta$. Applying Lemma-1, we are done. Q.E.D.

With $\rho = \delta = 1$, and $p = 1$, we get DePrima's (1974) result; and with $\rho = \delta = 2$, and $p = 1$, we obtain a result of Khasbardar and Thakare (1978)

We now deal with essential similarities involving ρ-oid operators. In fact, we have the following:

*Theorem* 2 — Let $T$ be an invertible operator such that

(i) $ST^* = T^{-p}S + K$, where $K$ is compact $0 \notin W_e(S)$, the essential numerical range of $S$ and $p$ ($\neq -1$) is some integer,

(ii) $T$ is $\rho$-oid, $T^{-1}$ is $\delta$-oid ($\rho, \delta \geqslant 1$),

(iii) $\sigma_{00}(T)$, the set of isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity is empty. Then $T$ is unitary.

PROOF : By applying Corollory 2 of Patel (1974) one concludes that $\sigma(T)$ lies on the unit circle $U$ with centre at the origin, and hence $r(T) = 1$. Since $T$ is $\rho$-oid, we have $w_\rho(T) = r(T) = 1$. Thus $T \in C_\rho$.

As $\sigma(T^{-1}) = \sigma(T)^{-1}$, it follows that $\sigma(T^{-1}) \subset U$ and $r(T^{-1}) = 1$. Since $T^{-1}$ is $\delta$-oid, $w_\delta(T^{-1}) = r(T^{-1}) = 1$ from which it follows that $T^{-1} \in C_\delta$. By Lemma 1, we reach the desired conclusion. Q.E.D.

Above considerations coupled with a result of Patel and Gupta (1975) lead us to analogous result for $M_\rho$-operators. Such a result could also be obtained for reduction $M_\rho$-operators essentially by the arguments of Patel (1974) and Gupta (1976). In the same vein one has the following modification of Sheth's (1969) result:

"If $T$ is convexoid and satisfies $ST^* = TS + K$, with compact $K$, $0 \notin W_e(S)$ and $\sigma_{00}(T) = \phi$, then $T$ is self-adjoint."

This motivates us to propose the following:

*Problem* — Is a $\rho$-convexoid operator (i.e. an operator $T \in \beta(H)$ with

$$W_\rho(T) = \text{conv. } \sigma(T))$$

with real spectrum necessarily self-adjoint; obviously for $\rho > 2$?

An affirmative answer to this result will yield a modification of Sheth's result mentioned above wherein convexoid will be replaced by $\rho$-convexoid.

Furuta and Nakamoto (1969) show that a contraction $T$ with $T^k = I$ is unitary. Further, they improve this result for an operator $T$ with $w(T) \leqslant 1$ instead of $\| T \| \leqslant 1$. We now obtain a generalization of this result.

*Theorem* 3 — If $T \in \beta(H)$ with $T^k = I$, where $k$ is some integer and $w_\rho(T) \leqslant 1$ ($\rho \geqslant 1$), then $T$ is unitary.

PROOF : Since $\sigma(T^k) = \sigma(T)^k = \{1\}$, $\sigma(T) \subset U$ and $T^{-1}$ exists with $T^{-1} = T^{k-1}$. Thus $w_\rho(T^{k-1}) \leqslant w_\rho(T)^{k-1} \leqslant 1$; i.e. $T^{-1} \in C_\rho$. As $T \in C_\rho$, the result follows from Lemma 1.

*Corollary* — Let $T$ be $\rho$-oid ($\rho \geqslant 1$) operator such that $T^k = I$, where $k$ is some integer. Then $T$ is unitary.

PROOF: As in the proof of Theorem 3, we have $r(T) = 1$. Further,

$$r(T) = w_\rho(T) = 1$$

and we are through. Q.E.D.

## 3. Conditions Implying Normality

Douglas and Rosenthal (1968) showed that an operator $T$ on $H$ is normal if and only if every quadratic polynomial in $T$ and $T^*$ is normaloid. Berberian (1970) extended this result for convexoid quadratic polynomial. Both these results can be extended.

Assume that every quadratic polynomial in $T$ and $T^*$ (i.e. the operator of the form

$$Q = aT^2 + bTT^* + cT^*T + dT^{*2} + eT + fT^* + gI;\ a, b, \ldots$$

are scalars) is spectraloid. Then for any complex $\alpha$,

$$Q - \alpha I = aT^2 + bTT^* + cT^*T + dT^{*2} + eT + fT^* + g'I$$

turns out to be spectraloid in view of the assumption. By applying often used characterization due to Furuta and Nakamoto (1971) (namely: "an operator $T$ is convexoid if and only if $T - \alpha I$ is spectraloid for each complex $\alpha$) we conclude that $Q$ is convexoid, i.e. every quadratic polynomial in $T$ and $T^*$ is convexoid Thus one obtains the following:

*Theorem* 4 — For any operator $T$ the following statements are equivalent:

(i) $T$ is normal;

(ii) every quadratic polynomial in $T$ and $T^*$ is convexoid;

(iii) every quadratic polynomial in $T$ and $T^*$ is normaloid;

(iv) every quadratic polynomial in $T$ and $T^*$ is spectraloid.

These considerations embolden one to pose the following:

*Problem* 2 — (i) If every quadratic polynomial in $T$ and $T^*$ is ρ-oid, need, $T$ be normal?

(ii) If every quadratic polynomial in $T$ and $T^*$ is ρ-convexoid, does it follow that $T$ is normal?

Let $P$ be some property of operators. We say that an operation $T$ is reduction-$P$ if the restriction of $T$ to every invariant subspace of $T$ has property $P$. We now obtain a generalization of a result of Constantin (preprint).

*Theorem* 5 — Let $T \in \beta(H)$ satisfying

(i) $T$ is restriction-convexoid;

(ii) $T$ is reduced by each of its eigenspaces; and

(iii) $T = S^{-1}A^p S + K$, where $\sigma(A)$ is real, $K$ is compact and $p$ is some non-negative integer.

Then $T$ is normal.

PROOF: From the hypothesis we infer that Weyl's theorem holds for $T$. This observation essentially follows from Berberian (1969 and 1970)]. Thus the Weyl spectrum $\sigma_w(T) = \sigma(T) - \sigma_{00}(T)$. Since the Weyl spectrum is preserved under similarity (see Istratescu 1971) and also remains invariant under compact perturbation, we have

$$\sigma_w(T) = \sigma_w(S^{-1}A^p S + K) = \sigma_w(S^{-1}A^p S) = \sigma_w(A^p) \subset \sigma(A)^p.$$

From this it is clear that $\sigma_w(T)$ is real. Let $T_1 = T \mid_{H_1}$ be the restriction of $T$ to the subspace $H_1$ generated by eigenvectors corresponding to eigenvalues $\lambda_0 \in \sigma_{00}(T)$. Let $H_2 = H_1^{\perp}$ and $T_2 = T \mid_{H_2}$. Then we obtain $H = H_1 \oplus H_2$. Since $T$ is reduced by each of its eigenspaces, we conclude that $T_1$ is normal. Again $\sigma(T_2) = \sigma_w(T)$ is real and hence $T_2$ is self-adjoint; which shows that $T$ is normal. Q.E.D.

## ACKNOWLEDGEMENT

The authors are grateful to the referee for making several suggestions.

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