*Indian J. pure appl. Math.*, **11**(9) : 1194–1201, September 1980

# SATURATION OF LOCAL APPROXIMATION BY LINEAR POSITIVE OPERATORS OF JACKSON TYPE*

A. WAFI

*Department of Mathematics, Aligarh Muslim University, Aligarh* 202001

(*Received* 24 *November* 1978; *after revision* 5 *November* 1979)

Suzuki (1965) proved local saturation theorems for the operators $L_n(f; x)$. In the present paper we have shown that analogous theorems hold true if we consider the linear positive operators of Jackson type i.e. $L_{ns-s}(f; x)$.

§1. Let $f(x)$ be an integrable function in $(-\pi, \pi)$ and periodic with period $2\pi$ and let its Fourier series be

$$S[f] \equiv \tfrac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{\infty} A_k(x). \qquad \ldots(1.1)$$

Let us consider the family of linear operators

$$L_n(f; x) \sum_{k=0}^{\infty} g_k^{(n)} A_k(x) \qquad \ldots(1.2)$$

where $g_k^{(n)}, k = 0, 1, 2, \ldots$ $(g_0^{(n)} = 1)$ are summating functions.

If there are a positive nonincreasing function $\phi(n)$ and a constant $K$ of functions such that

$$\| f(x) - L_n(f; x) \| = o(\phi(n))$$

if and only if $f(x)$ is constant,

$$\| f(x) - L_n(f; x) \| = O(\phi(n))$$

if and only if $f(x)$ belongs to class $K$;

then it is said that this method of approximation is saturated with the order $\phi(n)$ and the class $K$.

The above definition was given by Favard (1949) and he proposed to determine the order and class of saturation for various summation methods. Since then a number of contributions have been made and the problem was also extended to more

*This work was supported by Grant No. 7/112(537)/76–EMR–I of the Council of Scientific and Industrial Research of India.

general approximation processes (Alexits 1944, Butzer 1956, Favard 1957, Sunouchi and Watari 1958, Suzuki 1965, Zamansky 1949, etc).

Let

$$U_n(t) = \tfrac{1}{2} + \sum_{k=1}^{n} g_k^{(n)} \cos kt \geqslant 0, \ (-\pi \leqslant t \leqslant \pi)$$

be a nonnegative trigonometric polynomial of degree $n$. Then the linear positive operator (1.2), associated with $U_n(t)$, can be represented by

$$L_n(f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t)\, U_n(t)\, dt. \qquad \text{...(1.3)}$$

For the operator of this type Suzuki (1965) proved the following saturation theorems.

*Theorem* 1.1 — If

$$\| L_n(f; x) - f(x) \|_{(a,b)} = o\,(1 - g_1^{(n)}),$$

then $f(x)$ is a linear function in $[c, d]$, where $[c, d]$ is any fixed interval of $[a, b]$.

*Theorem* 1.2 — If

$$\| L_n(f; x) - f(x) \| = O(1 - g_1^{(n)}),$$

then $f'(x)$ belongs to the class Lip $(1, p; c, d)$.

*Theorem* 1.3 — If $f'(x) \in$ Lip $(1, p; a, b)$ and $L_n(t^2, 0) = O(1 - g_1^{(n)})$, then

$$\| L_n(f; x) - f(x) \|_{(c,d)} = O\,(1 - g_1^{(n)}).$$

§2. Schurer (1965) gave an example of a sequence of positive linear operators of which the Féjer and Jackson operators are particular cases. Here we shall determine the order of saturation and its class in the local approximation by a special form of the Jackson type operators $L_{ns-s} (n = 1, 2, \ldots)$ which is defined by

$$L_{ns-s}(f; x) = \frac{1}{A_{ns-s}} \int_{-\pi}^{\pi} f(x+t) \left( \frac{\sin \frac{1}{2} nt}{\sin \frac{1}{2} t} \right)^{2s} dt \qquad \text{...(2.1)}$$

where

$$A_{ns-s} = \int_{-\pi}^{\pi} \left( \frac{\sin \frac{1}{2} nt}{\sin \frac{1}{2} t} \right)^{2s} dt \ \ (n = 1, 2, \ldots).$$

Here $s$ is an arbitrary but fixed positive integer; the subscript $ns - s$ denotes that $\left(\frac{\sin \frac{1}{2} nt}{\sin \frac{1}{2} t}\right)^{2s}$ is an even trigonometric polynomial of degree $ns - s$.

The sequence of the form (2.1) is a special case of the sequence of operators (1.3) because

$$\left(\frac{\sin \frac{1}{2} nt}{\sin \frac{1}{2} t}\right)^{2s} = \tfrac{1}{6}\rho_0^{(ns-s)} + \tfrac{1}{3} \sum_{k=1}^{ns-s} \rho_k^{(ns-s)} \cos kt \geqslant 0 \quad (-\pi \leqslant t \leqslant \pi). \qquad \ldots(2.2)$$

In case $s = 1$ and $s = 2$, we have well-known Féjer and Jackson operators respectively:

$$L_{n-1}(f; x) = \frac{1}{2\pi n} \int_{-\pi}^{\pi} f(x+t) \left(\frac{\sin \frac{1}{2} nt}{\sin \frac{1}{2} t}\right)^2 dt$$

$$L_{2n-2}(f; x) = \frac{1}{2\pi(2n^3 + n)} \int_{-\pi}^{\pi} f(x+t) \left(\frac{\sin \frac{1}{2} nt}{\sin \frac{1}{2} t}\right)^4 dt.$$

Let us consider the family of linear positive operators

$$L_{ns-s}(f; x) = \sum_{k=0}^{\infty} \rho_k^{(ns-s)} A_k(x) \qquad \ldots(2.3)$$

where $\rho_k^{(ns-s)} = O(n^{2s-1}),\ k = 0, 1, 2, \ldots\ (\rho_0^{(ns-s)} \geqslant C_s n^{2s-1})$

are summating functions.

Since $\dfrac{\rho_k^{(ns-s)}}{\rho_0^{(ns-s)}} \to 1$ as $n \to \infty$ and $\rho_k^{(ns-s)}$ is a polynomial of degree $2s - 1$,

we have

$$\lim_{n\to\infty} \frac{\rho_k^{(ns-s)}}{1 - \rho_1^{(ns-s)}} = k^2 \quad (k = 1, 2, \ldots). \qquad \ldots(2.4)$$

Suppose that the linear positive operator (2.3) can be represented as follows:

$$L_{ns-s}(f; x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t)\, u_n(t)\, dt$$

where $u_n(t)$ is a kernel

$$u_n(t) = \tfrac{1}{2} + \sum_{k=1}^{ns-s} \frac{\rho_k^{(ns-s)}}{\rho_0^{(ns-s)}} \cos kt \geqslant 0$$

$$= \frac{1}{A_{ns-s}} \left( \frac{\sin \frac{1}{2} nt}{\sin \frac{1}{2} t} \right)^{2s}$$

with

$$\frac{1}{\pi} \int_{-\pi}^{\pi} u_n(t)\, dt = 1 \quad (-\pi \leqslant t \leqslant \pi).$$

§3. Throughout the paper the norm should be taken with respect to variable $x$, and the subscript $p (1 \leqslant p \leqslant \infty)$ to $L^p$ norm will be generally omitted. Another convention is that the space $C$ is meant by the notation $L^\infty$, and the interval $[c, d]$ is an arbitrary fixed subinterval of the given interval $[a, b]$ which is situated in $(-\pi, \pi)$. Also, let us write

$$\| L_{ns-s}(f; x) - f(x) \|_{(a,b)}$$

$$= \begin{cases} \left( \int_a^b | L_{ns-s}(f; x) - f(x) |^p \, dx \right)^{1/p} & (1 \leqslant p < \infty) \\ \max\limits_{x \in [a,b]} | L_{ns-s}(f; x) - f(x) | & (p = \infty), \end{cases}$$

and

$$\text{Lip}\,(1, p; a, b) = \{ f(x) : \sup_{|h| \leqslant \delta} \| f(x+h) - f(x) \|_{(a,b)} = O(\delta) \}.$$

§4. We establish the following results which are similar to Theorems 1.1, 1.2 and 1.3 of Suzuki (1965). We remark that this method is saturated locally; the order of saturation is $n^{-2}$ and the class of saturation $\{ f \in L^p : f' \in \text{Lip}\,(1, p) \}$. Again note that no extra assumptions are involved in proving our results for $L_{ns-s}(f; x)$.

Our results are:

*Theorem* 4.1 — For $f(x) \in L(-\pi, \pi) \cap L^p(a, b)$ $(1 \leqslant p \leqslant \infty)$, if

$$\| L_{ns-s}(f; x) - f(x) \|_{(a,b)} = o(n^{-2}) \qquad \text{...(4.1)}$$

then $f(x)$ is a linear function in $[c, d]$.

*Theorem* 4.2 — For $f(x) \in L(-\pi, \pi) \cap L^p(a, b)$ $(1 \leqslant p \leqslant \infty)$, if

$$\| L_{ns-s}(f; x) - f(x) \|_{(a,b)} = O(n^{-2}) \qquad \text{...(4.2)}$$

then $f'(x)$ belongs to Lip $(1, p; c, d)$.

*Theorem* 4.3 — If $f(x)$ belongs to the spaces $L(-\pi, \pi) \cap L^p(a, b)$ $(1 \leqslant p \leqslant \infty)$, and $f'(x)$ belongs to the class Lip $(1, p; a, b)$, then

$$\| L_{ns-s}(f; x) - f(x) \|_{(c,d)} = O(n^{-2}).$$

§5. Now, we start out with some well-known results that may be found in Sunouchi (1962) which will be needed in proving our theorems. His results read as follows:

*Theorem I* — A necessary and sufficient condition for $f''(x)$ to exist and belong to the class $B^*$ over $(a, b)$ is the uniform boundedness of $\sigma_m^2 [x, S'']$ over $[a, b]$, where $\sigma_m^2 [x, S'']$ means the $(C, 2)$-means of the second derived series of (1.1).

*Theorem II* — A necessary and sufficient condition for $f''(x)$ to exist and belong to the class $L^p (p > 1)$ over $(a, b)$ is

$$\int_a^b | \sigma_m^2 [x, S''] |^p \, dx = O(1).$$

*Theorem III* — A necessary and sufficient condition for $f'(x)$ to exist and belong to the class $BV$ over $(a, b)$ is

$$\int_a^b | \sigma_m^1 [x, S''] | \, dx = O(1).$$

*Proofs of Theorems* 4.1 *and* 4.2 — The proofs of Theorems 4.1 and 4.2 are almost the same. So we shall only give the proof of Theorem 4.2 with respect to $(C)$-norm. The proof of Theorems 4.1 and 4.2 in the $L^p$ space are analogous to the $(C)$-space.

Since

$$L_{ns-s}(f; x) - f(x) = O(1 - \rho_1^{(ns-s)}) = O(n^{-2}),$$

uniformly over $(a, b)$, we have

$$\sigma_m^2 \left[ x, \frac{1}{1 - \rho_1^{(ns-s)}}, \{L_{ns-s}(f; x) - f(x)\} \right] = O(1)$$

for every $m$ and uniformly in $x$ on any fixed subinterval of $[a, b]$, i.e. on $[c, d]$ because

$$\frac{1}{1 - \rho_1^{(ns-s)}} \{L_{ns-s}(f; x) - f(x)\} \sim \sum_{k=0}^{\infty} \frac{1 - \rho_k^{(ns-s)}}{1 - \rho_1^{(ns-s)}} A_k(x)$$

---

*$f''$ is the Fourier series of bounded function $(p = \infty)$.

(see Zygmund 1959, Theorem 9.20, p. 367). Letting $n \to \infty$, we get by (2.4)

$$\sigma_m^2 \left[x, \sum_{k=0}^{\infty} k^2 A_k(x)\right] = O(1)$$

hence we have $f''(x) \in B$ in $[c, d]$ from Theorem I.

*Remark*: We have only to apply Theorem II or III in order to verify the facts (4.1) and (4.2) in $L^p$ space ($p > 1$).

*Proof of Theorem* 4.3 — For the sake of simplicity, we shall only give the proof with respect to $L^p$-norm ($1 \leqslant p < \infty$). The case of $C$-norm may be treated by similar method.

Let us write $\delta = \min(c - a, b - d)$. By generalized Minkowski inequality. we have

$$\left(\int_c^d |L_{n_s - s}(f; x) - f(x)|^p dx\right)^{1/p} = \frac{1}{2\pi} \left\{\int_c^d \left|\int_0^\pi [f(x+t) + f(x-t) - 2f(x)]\, u_n(t)\, dt\right|^p dx\right\}^{1/p}$$

$$= \frac{1}{2\pi} \left\{\int_c^d \left|\left(\int_0^\delta + \int_\delta^\pi\right)[f(x+t) + f(x-t) - 2f(x)]\, u_n(t)\, dt\right|^p dx\right\}^{1/p}$$

$$\leqslant \frac{1}{2\pi} \left\{\int_c^d \left|\int_0^\delta [f(x+t) + f(x-t) - 2f(x)]\, u_n(t)\, dt\right|^p dx\right\}^{1/p}$$

$$+ \frac{1}{2\pi} \left\{\int_c^d \left|\int_\delta^\pi [f(x+t) + f(x-t) - 2f(x)]\, u_n(t)\, dt\right|^p dx\right\}^{1/p}$$

$$= I_1 + I_2, \text{ say.}$$

$$I_1 = \frac{1}{2\pi} \left\{\int_c^d \left|\int_0^\delta [f(x+t) + f(x-t) - 2f(x)]\, u_n(t)\, dt\right|^p dx\right\}^{1/p}$$

$$\leqslant \frac{1}{2\pi} \int_0^\delta u_n(t)\, dt \left(\int_c^d |f(x+t) + f(x-t) - 2f(x)|^p dx\right)^{1/p}$$

$$\leqslant \frac{1}{2\pi} \int_0^\delta u_n(t)\, dt\, w_p(|t|)$$

(*equation continued on p.* 1200)

$$= \frac{1}{2\pi}\int_0^{\delta} u_n(t)\, dt\, w_p\left(n\,|\,t\,|\,\frac{1}{n}\right)$$

$$\leqslant \frac{1}{2\pi}\int_0^{\delta} u_n(t)\, dt(n\,|\,t\,|+1)\, w_p\left(\frac{1}{n}\right)$$

$$\leqslant \frac{1}{2\pi} w_p\left(\frac{1}{n}\right)\int_0^{\pi} (n\,|\,t\,|+1)\, u_n(t)\, dt$$

$$= \frac{1}{2\pi} w_p\left(\frac{1}{n}\right)\left[\int_0^{\pi} u_n(t)\, dt + n\int_0^{\pi} |\,t\,|\, u_n(t)\, dt\right]$$

$$= K_1 w_p\left(\frac{1}{n}\right)\left[1 + n\int_0^{\pi} |\,t\,|\, u_n(t)\, dt\right]$$

$$\leqslant K_2 w_p\left(\frac{1}{n}\right) \qquad \text{(following Korovkin 1960, p. 71)}$$

$$= O(n^{-2}) \qquad \ldots(5.1)$$

where $w_p(\delta)$ is the modulus of continuity of $f(x)$ in $L^p(c, d)$.

$$I_2 = \frac{1}{2\pi}\left\{\int_c^d \left|\int_\delta^\pi [f(x+t) + f(x-t) - 2(x)]\, u_n(t)\, dt\right|^p dx\right\}^{1/p}$$

$$\leqslant \frac{1}{2\pi}\left\{\int_c^d dx\left(\int_\delta^\pi |\,f(x+t) + f(x-t) - 2f(x)\,|\, u_n(t)\, dt\right)^p\right\}^{1/p}$$

$$\leqslant \frac{K_3}{A_{ns-s}}\left\{\int_c^d dx\left(\int_\delta^\pi |\,f(x+t) + f(x-t) - 2f(x)\,|\left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^{2s} dt\right)^p\right\}^{1/p}$$

Since

$$\left|\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right| \leqslant n$$

for all $n$ and $t$ and

$$\frac{\sin t}{t} \geqslant \frac{\sin \pi/2}{\pi/2} = \frac{2}{\pi},\ t \in [0, \pi/2].$$

And hence we have

$$\left|\frac{\sin \frac{1}{2} nt}{\sin \frac{1}{2} t}\right| \leqslant \frac{1}{|\sin \frac{1}{2} t|} \leqslant \pi/t.$$

In view of the above inequalities, we have

$$I_2 \leqslant \frac{K_4}{A_{ns-s}} \left\{\int_c^d dx \left(\int_\delta^\pi | f(x+t) + f(x-t) - 2f(x) | \, t^{-2s} dt\right)^p\right\}^{1/p}$$

$$\leqslant K_5 w_p\left(\frac{1}{n}\right) \qquad \text{(following Suzuki 1965)}$$

$$= O(n^{-2}). \qquad \ldots(5.2)$$

Hence by (5.1) and (5.2), we have

$$\| L_{ns-s}(f; x) - f(x) \|_{(c,d)} = O(n^{-2}).$$

Thus the approximation method is saturated locally for space $L^p (1 \leqslant p \leqslant \infty)$, its order of saturation is $n^{-2}$ and its class of saturation is $\{f : f' \in \text{Lip}\,(1, p)\}$.

## Acknowledgement

The author wishes to express his thanks to Prof. S. Umar and the referee for their helpful suggestions.

## References

Alexits, G. (1944). On the order of approximation of Féjer means. *Hung. Acta Math.*, **3**, 20–25.

Butzer, P. L. (1956). On the singular integrals of de la Vallée Poussin. *Arch. Math.*, **7**, 295–309.

Favard, J. (1949). Sur l' approximation des fonction d' une variable réelle; Analyse Harmonique. *Coll. Internat. du centre Nat. Rech, Sci. Paris*, **5**, 97–110.

———— (1957). Sur la saturation des procédés de sommation. *J. Math.*, **36**, 359–72.

Korovkin, P. P. (1960). Linear Operators and Approximation Theory. Hindustan Publishing Corp., Delhi.

Schurer, F. (1965). On linear positive operators in approximation theory. Thesis, Delft.

Sunouchi, G. (1962). Local operators on trigonometric series. *Trans. Am. math. Soc.*, **104**, 457–61.

Sunouchi, G., and Watari, C. (1958). On determination of the class of saturation in the theory of approximation of functions II. *Tohoku Math. J.*, **11**, 480–88.

Suzuki, Y. (1965). Saturation of local approximation by linear positive operators. *Tôhoku Math. J.*, **17**, 210–21.

Zamansky, M. (1949). Classes de saturation de certains procédes d'approximation de series de Fourier des fonctions. *Ann. Sci. Ecole Normale Sup.*, **66**, 19–93.

Zygmund, A. (1959). Trigonometric Series, Vol. I Cambridge University Press, Cambridge.