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SPHERICALLY SYMMETRIC SPACE-TIMES IN BI-METRIC RELATIVITY THEORY—I

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The various forms of spherically symmetric space-times in bi-metric relativity theory of Rosen are considered. The vacuum space-time solutions are then derived.

1. INTRODUCTION

The spherical symmetry has its own importance in general relativity theory by virtue of its comparative simplicity. Many noteworthy space-times : Schwarzschild solutions (exterior and interior), the Robertson-Walker model of the expanding universe etc. are all spherically symmetric (SS). Takeno (1966) has brilliantly brought out the new mathematical aspects of SS space-times. As the mathematical problems associated with spherical symmetry are so far being exhausted, we have taken up a project to discuss some aspects of it with reference to the gravitation theory proposed by Rosen (1977). This new theory is widely known as 'bi-metric relativity theory', as it is based on two metric tensors g_{ij} and f_{ij}*. The first metric tensor describes the curved space-time and thereby the gravitational field. The second metric tensor refers to the flat space-time, whose curvature tensor derived from f_{ij} vanishes, and describes the inertial forces associated with the acceleration of the frame of reference. Thus at every point of a space-time there are two metrics:

$$ds^2 = g_{ij}dx^i dx^j \qquad ...(1.1)$$

$$d\sigma^2 = f_{ij}dx^i dx^j. \qquad ...(1.2)$$

Then is defined a covariant differentiation based on f_{ij} — f differentiation — denoted by a bar (|) and obeys all the rules of ordinary differentiation but with exception

$$f_{ij\,|\,k} = 0.$$

The field equations of Rosen's theory are

$$K^j_i = N^j_i - \tfrac{1}{2}\,\delta^j_i\, N = -8\pi k T^j_i \qquad ...(1.3)$$

where $\quad N^j_i = \tfrac{1}{2} f^{pr}(g^{sj}g_{si\,|\,p})_{\,|\,r}$

*Rosen uses γ_{ij}.

$$N = N^i_i$$

$$k = \sqrt{g/f}, \; g = \det g_{ij} \text{ and } f = \det f_{ij}.$$

Choosing f_{ij} as the Lorentz metric, the f-derivative becomes the ordinary partial derivative.

The theory has a simpler mathematical structure than that of the general relativity. Moreover the theory agrees with general relativity up to the accuracy of the observation made till now. However, the bi-metric relativity does not account for black holes, a creation of general relativity. In this connection we append the remarks of Rosen (1977) : 'If these black holes are found in nature, this will be a great success of general relativity. Since there is no convincing evidence at present that black holes exist, one can take the standpoint that they represent a breakdown of the usual concepts of space-time and are therefore something unphysical'. Of course if the black holes are discovered in near future, one has to keep his mind open to modify the bi-metric relativity in favour of general relativity.

2. The SS Line Element

Takeno (1966) has rigorously established that the most general form of the SS line element is

$$ds^2 = -Adr^2 - Bd\Sigma^2 + Cdt^2 + 2Ddr\,dt \qquad \text{...(2.1)}$$

where A, B, C and D are functions of r and t and $d\Sigma^2 = d\theta^2 + \sin^2\theta\, d\phi^2$.

The flat space-time corresponding to (2.1) is

$$d\sigma^2 = -dr^2 - r^2 d\Sigma^2 + dt^2. \qquad \text{...(2.2)}$$

It is easy to see that

$$g = -mB^2 \sin^2\theta, \; m = AC + D^2$$

$$f = -r^4 \sin^2\theta, \; k = (mB^2/r^4)^{1/2}.$$

The non-vanishing f-Christoffel symbols of second kind are

$$\Gamma^2_{12} = \Gamma^3_{13} = 1/r, \; \Gamma^1_{22} = -r, \; \Gamma^1_{33} = -r\sin^2\theta,$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta, \; \Gamma^3_{23} = \cot\theta.$$

The straightforward calculations yield

$$\begin{aligned} 4K^1_1 = & -r^{-2}\,[r^2(CA' - AC')/m]' + [(C\dot{A} - A\dot{C})/m]^{\cdot} \\ & + 4\,[(A/B) - (BC/mr^4)] - 2(\dot{B}/B)^{\cdot} \\ & + 2r^{-1}\,[r \log (B/r^2)]'' \end{aligned} \qquad \text{...(2.3)}$$

$$4K_2^2 = 4K_3^3 = -2\,[(A/B) - (BC/mr^4)] + r^{-1}\,[(r \log m)'' - (r \log m)^{\cdot\cdot}] \qquad \ldots(2.4)$$

$$4K_4^4 = -r^{-2}\,[r^2(AC' - CA')/m]' + [(A\dot{C} - C\dot{A})/m]^{\cdot} - 2(\dot{B}/B)^{\cdot} + 2r^{-1}\,[r \log (B/r^2)]'' \qquad \ldots(2.5)$$

$$K_1^4 = N_1^4 \text{ and } K_4^1 = N_4^1, \text{ etc.} \qquad \ldots(2.6)$$

All other K_i^j are zero. The prime ($'$) and the dot ($^{\cdot}$) denote the partial derivative w.r.t. r and t respectively.

3. Exterior Solutions

In this section are discussed the empty space-time solutions, corresponding to $T_i^j = 0$ i.e. $N_i^j = 0$, in some special forms of the line-element derived from (2.1).

(a) *Orthogonal Form*

Takeno (1966) has shown that the line-element (2.1) can be put into the orthogonal form

$$ds^2 = -Adr^2 - Bd\Sigma^2 + Cdt^2 \qquad \ldots(3.1)$$

by a suitable coordinate transformation.

Takeno calls (3.1) a SS line-element in a narrow sense. The field equations $N_i^j = 0$ or $K_i^j = 0$ now yield

$$-(r \log C)'' + (r \log C)^{\cdot\cdot} = 0 \qquad \ldots(3.2)$$

and

$$-[r \log (AB^2/r^4)]'' + [r \log (AB^2/r^4)]^{\cdot\cdot} = 0. \qquad \ldots(3.3)$$

From eqns. (3.2) and (3.3), we obtain

$$C = \exp\,[r^{-1}\,\{F(t + r) + G(t - r)\}] \qquad \ldots(3.4)$$

and

$$AB^2 = r^4 \exp\,[r^{-1}\,\{H(t + r) + I(t - r)\}] \qquad \ldots(3.5)$$

where F, G, etc. are arbitrary functions of their arguments $(t + r)$ and $(t - r)$ respectively.

Choosing $B = r^2A$, an isotropic line-element

$$ds^2 = -A(dr^2 + r^2d\Sigma^2) + Cdt^2 \qquad \ldots(3.6)$$

is obtained.

If the functions F, G, H and I are suitably adjusted such that $F = H$ and $G = I$, the form (3.6) gives a conformally flat space-time

$$ds^2 = e^{(F+G)/r}(-dr^2 - r^2 d\Sigma^2 + dt^2). \quad ...(3.7)$$

Furthermore taking $B = r^2$ in (3.1), a SS line-element in a curvature coordinate is obtained and the interpretation of 'r' stands as given by Synge (1971). In such a case one of the field equations ($N_2^2 = 0$) gives

$$A = \pm 1.$$

But A must be positive and hence the SS line-element becomes

$$ds^2 = -dr^2 - r^2 d\Sigma^2 + e^{(F+G)/r} dt^2. \quad ...(3.8)$$

From the above discussion it is certain that the well-known Birkhoff's theorem in general relativity is not valid in bi-metric relativity theory. However, restricting the forms of arbitrary functions F and G and then defining a new time, one can always bring (3.8) into a static form. But in general this is not true i.e. the exterior SS field, in a curvature coordinate, need not be static.

We shall now give some thought to the static SS solutions of bi-metric relativity.

(a.1) *The purely static solution with $B = r^2$* — Here the field equations are simplified and they yield

$$A = 1, C = e^{-\alpha/r},$$

where α is a constant of integration. Then we get

$$ds^2 = -dr^2 - r^2 d\Sigma^2 + e^{-\alpha/r} dt^2. \quad ...(3.9)$$

This is a new static solution and is not covered by the solution obtained by Rosen (1977). However, if one equates the secondary mass M', defined by Rosen (1977) to zero, the line-element (3.9) can be identified with Rosen's solution by interpreting our α as double the primary mass M. Moreover by definition

$$M' = 4\pi \int_0^R k(\epsilon - p)\, r^2\, dr$$

where R is the radius of the stellar body, ϵ the density and p the pressure.

Now $M' = 0$ when $\epsilon = p$ or $\epsilon = 0 = p$. Thus retaining $T_i^j \neq 0$, the vanishing of M' imposes a condition ($\epsilon = p$) on a stellar structure. This condition, which is not other than equation of state, is for a stiff matter. Strictly speaking (3.9) cannot be deduced from Rosen's eqns. (2.6) and (2.7) unless the arbitrary constants of integration of (2.7) are all equated to zero.

Let us now take $\alpha = 2M$, where M can be interpreted as the primary mass of the central body which also equals the total energy of the body. This M corresponds to the Newtonian mass and determines the motion of a test particle at a large distance from the central body as remarked by Rosen (1977).

We shall now derive the planetary equation of motion in the field of (3.9) with $\alpha = 2M$.

For a freely moving particle in the field of (3.9), the three Euler-Lagrange equations for θ, ϕ and t associated with the variational problem are

$$\frac{d}{ds}\left(r^2 \frac{d\theta}{ds}\right) = r^2 \sin\theta \cos\theta \left(\frac{d\phi}{ds}\right)^2 \quad \text{...(3.10)}$$

$$\frac{d}{ds}\left(r^2 \sin^2\theta \frac{d\phi}{ds}\right) = 0 \quad \text{...(3.11)}$$

$$\frac{d}{ds}\left(e^{-2M/r} \frac{dt}{ds}\right) = 0. \quad \text{...(3.12)}$$

By adjusting the initial conditions we find from (3.10) - (3.12) that the particle continues to move in the plane $\theta = \pi/2$. Then the line-element (3.9) itself will yield the first integral of the motion as

$$1 = -\left(\frac{dr}{ds}\right)^2 - r^2\left(\frac{d\phi}{ds}\right)^2 + e^{-2M/r}\left(\frac{dt}{ds}\right)^2. \quad \text{...(3.13)}$$

From (3.11), we derive

$$r^2 \frac{d\phi}{ds} = H \text{ etc.}$$

The straightforward calculations then give the equation of a planetary motion as

$$\frac{d^2u}{d\phi^2} - M\left(\frac{du}{d\phi}\right)^2 + u = Mu^2 + \frac{M}{H^2} \quad \text{...(3.14)}$$

with $u = 1/r$.

The corresponding equation in general relativity is

$$\frac{d^2u}{d\phi^2} + u = 3mu^2 + m/h^2. \quad \text{...(3.15)}$$

For smaller values of $du/d\phi$ and $M = 3m$, $H^2 = 3h^2$, (3.14) identifies with (3.15). But then the perihelic shift of mercury will be only one-third of the value predicted by general relativity.

(a.2) *Static solution with $B = Ar^2$* — In this case the field equations give

$$ds^2 = -e^{\alpha/r}(dr^2 + r^2 d\Sigma^2) + e^{\beta/r}dt^2. \quad \text{...(3.16)}$$

Assigning suitable values to α and β, (3.16) becomes the space-time obtained by Rosen (1977). Here α and β refer to the primary mass and the secondary mass of the central body respectively.

(a.3) *Pseudo static solution* — Now consider the situations corresponding to

(i) $B = r^2 A(t)$ and $C = C(r)$

and (ii) $B = r^2 A(r)$ and $C = C(t)$.

In the first case, we achieve

$$ds^2 = -e^{\alpha t}(dr^2 + r^2 d\Sigma^2) + e^{\beta/r} dt^2 \qquad ...(3.17)$$

whereas in the second case

$$ds^2 = -e^{\alpha/r}(dr^2 + r^2 d\Sigma^2) + e^{\beta t} dt^2. \qquad ...(3.18)$$

Adjusting a time scale (3.18) can be written as

$$ds^2 = -e^{\alpha/r}(dr^2 + r^2 d\Sigma^2) + dt^2. \qquad ...(3.19)$$

(a.4) *Purely non-static solution* — Considering $C = C(t)$ and $B = r^2 T(t)$, the field equations yield

$$T = A = e^{\alpha t} \text{ and } C = e^{\gamma t}.$$

Again adjusting a time scale, we write

$$ds^2 = -e^{\alpha t}(dr^2 + r^2 d\Sigma^2) + dt^2. \qquad ...(3.20)$$

This corresponds to the de Sitter universe (see Tolman (1934)) of the general relativity theory. Moreover putting $\alpha = \gamma$, we have an expanding universe which is conformally flat.

(b) *D-form*

Equating A and C to zero, we arrive at

$$ds^2 = 2D\, dr\, dt - B d\Sigma^2 \qquad ...(3.21)$$

which is called a SS line-element in D-form. The coordinates used in this form are called 'null coordinates' and their geometrical significance is well explained by Synge (1971).

Here, for empty space-time, the field equations imply $B = 0 = D$. Therefore we arrive at an important theorem : 'SS vacuum solution of bi-metric relativity in the system of null coordinates is non-existent'.

From the consideration of the eigen values of N_i^j, a D-form is obtained in the next section.

4. Eigen Values of N_i^j

Denoting the eigen values of SS N_i^j by n, they are determined by the equation

$$\det (N_i^j - n\delta_i^j) = 0. \qquad \text{...(4.1)}$$

After simplification (4.1) gives

$$(N_2^2 - n)(N_3^3 - n)[n^2 - n(N_1^1 + N_4^4) + N_1^1 N_4^4 - N_1^4 N_4^1] = 0 \quad \text{...(4.2)}$$

establishing that the two of the principal invariants of N_i^j are equal (as $N_2^2 = N_3^3$). Therefore the form of n is (n_1, n_1, n_2, n_3) where $n_1 = N_2^2$ etc. The other two (n_2, n_3) are the solutions of the quadratic equation

$$n^2 - n(N_1^1 + N_4^4) + N_1^1 N_4^4 - N_1^4 N_4^1 = 0. \qquad \text{...(4.3)}$$

For the D-form $N_1^1 = N_4^4$ and hence (4.3) gives,

$$n^2 - 2nN_1^1 + (N_1^1)^2 - N_1^4 N_4^1 = 0. \qquad \text{...(4.4)}$$

The discriminant of the quadratic eqn. (4.4) is $-8/r^4$ and hence negative. Therefore n_2 and n_3 will be complex conjugate of each other i.e.

$$(n) \equiv (n_1, n_1, n_2, \bar{n}_2)$$

where $\bar{n}_2$ is the conjugate of n_2.

Taking n_2 s.t. $n_2 = -\bar{n}_2$ i.e. giving $N_1^1 = 0$, we get

$$D = \exp r^{-1} [F(t + r) + G(t - r)].$$

Furthermore, supposing $n_1 = 0$, B is determined by

$$-[r \log (B/r^2)]'' + [r \log (B/r^2)]^{\cdot\cdot} = 0$$

i.e., $$B = r^2 \exp r^{-1} [f(t + r) + g(t - r)].$$

Making the choice $F = f$, $G = g$, etc.

$$ds^2 = e^{(F+G)/r} [2dr\, dt - r^2 d\Sigma^2]. \qquad \text{...(4.5)}$$

We then arrive at the result: 'The SS line-element in the null coordinate system can be expressed in the form (4.5) when the principal invariants of N_i^j are of the form $(0, 0, n, -\bar{n})$'.

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