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SURFACE LINE SOURCES OVER A GENERALIZED THERMOELASTIC HALFSPACE

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Expressions for the displacement components together with the temperatures for both compressive as well as distortional line sources over an isotropic homogeneous generalized thermoelastic halfspace are obtained in what follows. These results are later applied to determine the correction in the displacement components and temperature when the free surface is slightly curved. As expected, these turn out to be integrals of linear expressions involving the displacements and temperatures over the plane-faced halfspace.

This paper dealing with surface line sources over a halfspace forms an extension of some of the earlier results in the theory of coupled thermoelasticity to generalized thermoelasticity (cf. Harinath 1975).

Suppose we consider an isotropic homogeneous perfectly elastic halfspace with a plane boundary made up of a heat conducting material of density ρ, thermal conductivity k, specific heat at constant strain s, and initially maintained at a constant temperature θ°. Let us set up a rectangular Cartesian coordinate system (x, y, z) with origin at the free surface in such a way that the solid halfspace is represented by $z \leqslant 0$ with $z = 0$ as the free surface. We assume that all the quantities are independent of the y-coordinate thereby reducing the problem considered into one of two-dimensional plane strain. The displacement vector **D** in terms of two potential functions Ω and χ may be expressed as:

$$\mathbf{D} = (u, 0, w) = \text{grad}\,\Omega + \text{curl}\,(0, \chi, 0); \frac{\partial \Omega}{\partial y} = 0, \text{div}\,(0, \chi, 0) = 0 \quad \ldots(1)$$

or equivalently as:

$$u = \frac{\partial \Omega}{\partial x} - \frac{\partial \chi}{\partial z}, \quad w = \frac{\partial \Omega}{\partial z} + \frac{\partial \chi}{\partial x} \quad \ldots(2)$$

where the potential functions Ω, χ and the temperature perturbation θ from θ° satisfy the partial differential equations:

$$\left.\begin{aligned}\rho\ddot{\Omega} &= \rho\alpha^2\left(\frac{\partial^2\Omega}{\partial x^2}+\frac{\partial^2\Omega}{\partial z^2}\right)-\gamma(\theta+\tau'\dot{\theta})\\ \ddot{\chi} &= \beta^2\left(\frac{\partial^2\chi}{\partial x^2}+\frac{\partial^2\chi}{\partial z^2}\right)\\ k\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial z^2}\right) &= \rho s(\dot{\theta}+\tau'\ddot{\theta})+\gamma\theta^\circ\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)[\dot{\Omega}+\tau'\ddot{\Omega}].\end{aligned}\right\} \quad ...(3)$$

α represents the isothermal compressional wave velocity, β the shear wave velocity, γ the ratio of the coefficient of thermal expansion to isothermal compressibility, τ' the relaxation time and a superimposed dot indicates differentiation with reference to the time t. Moreover, the second equation in (3), shows that shear waves are uninfluenced by the thermal terms.

In order to determine waves propagated along the x-direction, we assume that the solutions are proportional to exp $(ipt - ifx)$ where p denotes the frequency parameter and f denotes the wave number. As in Harinath (1976), solving (3) we obtain:

$$\left.\begin{aligned}\Omega &= \int_{-\infty}^{\infty} A(f)\,.\exp[g_1z+ipt-ifx]\,df\\ &\quad+\int_{-\infty}^{\infty} B(f)\,.\exp[g_2z+ipt-ifx]\,df\\ \chi &= \int_{-\infty}^{\infty} C(f)\,.\exp[g_3z+ipt-ifx]\,df\\ \gamma\tau\theta &= \rho(p^2-\alpha^2\zeta_1^2)\int_{-\infty}^{\infty} A(f)\exp[g_1z+ipt-ifx]\,df\\ &\quad+\rho(p^2-\alpha^2\zeta_2^2)\int_{-\infty}^{\infty} B(f)\,.\exp[g_2z+ipt-ifx]\,df\end{aligned}\right\} \quad ...(4)$$

wherein the unknown functions $A(f)$, $B(f)$, $C(f)$ have to be determined using the boundary conditions; the following notations are made use of in (4):

$$\left.\begin{aligned}&\epsilon=\frac{\gamma^2\theta^\circ}{\rho^2 s\alpha^2},\quad \tau=1+ip\tau',\quad \zeta_3^2=p^2/\beta^2\\ &\zeta_1^2,\zeta_2^2=\frac{1}{2k\alpha^2}\{kp^2-ip\rho s\tau\alpha^2(1+\epsilon\tau)\\ &\qquad\pm\sqrt{[kp^2-ip\rho s\tau\alpha^2(1+\epsilon\tau)]^2+4i\tau\rho sk\alpha^2p^3}\}\\ &g_j=\sqrt{f^2-\zeta_j^2},\quad \mathrm{Re}\,(g_j)\geqslant 0\ (j=1,2,3).\end{aligned}\right\} \quad ...(5)$$

The above solutions (4) are used to calculate the normal and shear stresses in the half-space given by the expressions:

$$\left.\begin{aligned}\sigma_{zz} &= \rho\alpha^2\left(\frac{\partial^2\Omega}{\partial x^2}+\frac{\partial^2\Omega}{\partial z^2}\right)+2\rho\beta^2\left(\frac{\partial^2\chi}{\partial x\partial z}-\frac{\partial^2\chi}{\partial x^2}\right)-\gamma\tau\theta\\ \sigma_{xz} &= \rho\beta^2\left(2\frac{\partial^2\Omega}{\partial x\partial z}+\frac{\partial^2\chi}{\partial x^2}-\frac{\partial^2\chi}{\partial z^2}\right).\end{aligned}\right\}\quad ...(6)$$

We now separately consider the compressive and distortional line sources over the halfspace.

Compressive line source — The boundary conditions across the free surface $z = 0$ in this case are

$$\sigma_{zz} = \rho\delta(x),\ \ \sigma_{xz} = 0,\ \frac{\partial\theta}{\partial z} + h\theta = 0$$

where $\delta(x)$ denotes the Dirac δ-function and h is a constant. These conditions in conjunction with eqns. (4) and (6) lead to the equations:

$$\left.\begin{aligned}&(2f^2\beta^2 - p^2)\,[A(f)+B(f)] + 2ipg_3\beta^2C(f) = 1/2\pi\\ &2if\beta^2\,\{g_1A(f)+g_2B(f)\} - (2f^2\beta^2-p^2)\,C(f) = 0\\ &(h-g_1)\,(p^2-\alpha^2\zeta_1^2)\,A(f) + (h-g_2)\,(p^2-\alpha^2\zeta_2^2)\,B(f) = 0\end{aligned}\right\}\quad ...(7)$$

which on solving yield

$$\left.\begin{aligned}A(f) &= (2f^2\beta^2-p^2)\,(h-g_2)\,(p^2-\alpha^2\zeta_2^2)/2\pi\Delta(f)\\ B(f) &= -\,(2f^2\beta^2-p^2)\,(h-g_1)\,(p^2-\alpha^2\zeta_1^2)/2\pi\Delta(f)\\ C(f) &= if\beta^2(g_1-g_2)\,[hp^2-\alpha^2\,\{h(f^2+g_1g_2)\\ &\qquad -(g_1+g_2)\,g_1g_2\}]/\pi\Delta(f)\end{aligned}\right\}\quad ...(8)$$

where

$$\Delta(f) = \begin{vmatrix} 2f^2\beta^2-p^2 & 2f^2\beta^2-p^2 & 2ipg_3\beta^2\\ 2ifg_1\beta^2 & 2ifg_2\beta^2 & -(2f^2\beta^2-p^2)\\ (h-g_1)(p^2-\alpha^2\zeta_1^2) & (h-g_2)(p^2-\alpha^2\zeta_2^2) & 0\end{vmatrix}$$

$$\equiv (h-g_2)\,(p^2-\alpha^2\zeta_2^2)\,[(2f^2\beta^2-p^2)^2-4g_1g_3f^2\beta^4]$$

$$-\,(h-g_1)\,(p^2-\alpha^2\zeta_1^2)\,[(2f^2\beta^2-p^2)^2-4g_2g_3f^2\beta^4].\quad ...(9)$$

Substituting eqns. (8) in eqns. (4) we get Ω, χ, θ. Since we are concerned with the surface displacements over the free surface $z = 0$, we denote these by $u_N(x, 0)$ and $w_N(x, 0)$. The temperature is denoted by $\theta_N(x, 0)$. These are due to the compressive line source produced by a normal force along the y-axis, and a calculation yields

$$u_N(x, 0) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \begin{Bmatrix} (2f^2\beta^2 - p^2)\,[(h - g_1)(p^2 - \alpha^2\zeta_1^2) - (h - g_2)(p^2 - \alpha^2\zeta_2^2)] \\ -2g_3\beta^2(g_1 - g_2)\,[hp^2 - h\alpha^2(f^2 + g_1g_2) + \alpha^2 g_1 g_2(g_1 + g_2)] \end{Bmatrix} \times \frac{f \exp(ipt - ifx)}{\Delta(f)}\, df$$

$$w_N(x, 0) = \frac{p^2}{2\pi} \int_{-\infty}^{\infty} (g_1 - g_2)\,[hp^2 - h\alpha^2(f^2 + g_1g_2) + \alpha^2 g_1 g_2(g_1 + g_2)] \times \frac{\exp(ipt - ifx)}{\Delta(f)}\, df \qquad \ldots(10)$$

$$\theta_N(x, 0) = \frac{(p^2 - \alpha^2\zeta_1^2)(p^2 - \alpha^2\zeta_2^2)}{2\pi\gamma\alpha^2\beta^4} \int_{-\infty}^{\infty} (2f^2\beta^2 - p^2)(g_1 - g_2) \times \frac{\exp(ipt - ifx)}{\Delta(f)}\, df. \qquad \ldots(11)$$

Distortional line source — In this case the boundary conditions across $z = 0$ are

$$\sigma_{zz} = 0,\ \sigma_{xz} = \rho\delta(x),\ \frac{\partial\theta}{\partial z} + h\theta = 0.$$

These conditions in conjunction with eqns. (4) and (6) lead to equations connecting $A(f)$, $B(f)$, $C(f)$ similar to (7) which on solving yield the solutions of Ω, χ, θ. Denoting the surface displacements across the surface by $u_T(x, 0)$, $w_T(x, 0)$ and the temperature by $\theta_T(x, 0)$, which are due to a distortional line source produced by a tangential force along the y-axis, a calculation yields:

$$u_T(x, 0) = \frac{p^2}{2\pi} \int_{-\infty}^{\infty} [(h - g_1)(p^2 - \alpha^2\zeta_1^2) - (h - g_2)(p^2 - \alpha^2\zeta_2^2)] \times \frac{g_3 \exp(ipt - ifx)}{\Delta(f)}\, df \qquad \ldots(12)$$

$$w_T(x, 0) = -u_N(x, 0)$$

$$\theta_T(x, 0) = \frac{i(p^2 - \alpha^2\zeta_1^2)(p^2 - \alpha^2\zeta_2^2)}{\pi\gamma\alpha^2\beta^2} \int_{-\infty}^{\infty} g_3(g_1 - g_2)\, \frac{f \exp(ipt - ifx)}{\Delta(f)}\, df. \qquad \ldots(13)$$

Equations (10) and (12) together yield the required displacement components while eqns. (11) and (13) yield the temperatures over the free surface $z = 0$. All these expressions are made use of in the application incorporated below.

Application — Suppose the free surface of the halfspace is slightly curved in the form of a cylinder with the y-axis along a generator, so that its equation is given by, $z = cm(x)$, where c is a small constant whose squares and higher powers are neglected and $m(x)$ is a function of x possessing a Fourier transform. Over the free surface, the displacements and temperatures may be divided into zero order terms and first order terms, the former yielding the results due to a plane-faced free surface and the latter yielding the effects of curvature. Since we are interested in the effects of curvature of the free surface, we denote the first order terms by $u_1(x, 0)$, $w_1(x, 0)$, $\theta_1(x, 0)$ and term these as the correction terms due to the curvature.

A lengthy calculation leads to the following integrals yielding the correction terms involving f_0, a root of $\triangle(f) = 0$:

$$u_1(x, 0) = \int_{-\infty}^{\infty} \begin{Bmatrix} -\rho\beta^2 w_T(x - f, 0)\, E_1(f_0) + \rho\beta^2 u_T(x - f, 0)\, E_2(f, f_0) \\ + \gamma p^2\alpha^2\theta_T(x - f, 0)\, E_3(f_0)/(p^2 - \alpha^2\zeta_1^2)\,(p^2 - \alpha^2\zeta_2^2) \end{Bmatrix}$$

$$\times m(f) \exp(ipt - if_0 f)\, df$$

$$+ i\rho\beta^2 E_4(f_0) \int_{-\infty}^{\infty} u_T(x - f, 0)\, m'(f) \exp(ipt - if_0 f)\, df \quad \text{...(14)}$$

$$w_1(x, 0) = \int_{-\infty}^{\infty} \begin{Bmatrix} \rho\beta^2 w_N(x - f, 0)\, E_1(f_0) - \rho\beta^2 u_N(x - f, 0)\, E_2(f, f_0) \\ -\gamma p^2\alpha^2\theta_N(x - f, 0)\, E_3(f_0)/(p^2 - \alpha^2\zeta_1^2)\,(p^2 - \alpha^2\zeta_2^2) \end{Bmatrix}$$

$$\times m(f) \exp(ipt - if_0 f)\, df$$

$$- i\rho\beta^2 E_4(f_0) \int_{-\infty}^{\infty} u_N(x - f, 0)\, m'(f) \exp(ipt - if_0 f)\, df \quad \text{...(15)}$$

$$\theta_1(x, 0) = \rho\beta^2 \int_{-\infty}^{\infty} [\theta_N(x - f, 0)\, E_1(f_0) + \theta_T(x - f, 0)\, E_2(f, f_0)]$$

$$\times m(f) \exp(ipt - if_0 f)\, df$$

$$+ i\rho\beta^2 E_4(f_0) \int_{-\infty}^{\infty} \theta_T(x - f, 0)\, m'(f) \exp(ipt - if_0 f)\, df$$

$$+ \frac{\rho E_3(f_0)}{2\pi\gamma\beta^4} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{Bmatrix} 4f^2\beta^4 g_3\, [g_2(p^2 - \alpha^2\zeta_1^2) - g_1(p^2 - \alpha^2\zeta_2^2)] \\ + \alpha^2(\zeta_1^2 - \zeta_2^2)\,(2f^2\beta^2 - p^2)^2 \end{Bmatrix}$$

$$\times \frac{\exp(ipt - if_0 f + if_0 q)}{\triangle(f)}\, m(q)\, dq\, df \quad \text{...(16)}$$

where the following notations have been employed:

$$(2f_0^2\beta^2 - p^2)(h - g_2^0)(p^2 - \alpha^2\zeta_2^2)\,E_1(f_0)$$
$$= p^4\beta^2(g_1^0 - g_2^0)\{hp^2 + h\alpha^2(f_0^2 + g_1^0 g_2^0) - \alpha^2 g_1^0 g_2^0 (g_1^0 + g_2^0)\}$$

$$(2f_0^2\beta^2 - p^2)(h - g_2^0)(p^2 - \alpha^2\zeta_2^2)\,E_2(f, f_0)$$
$$= -\,2i(g_1^0 - g_2^0)\{fE_2^*(f_0) - f_0E_2^{**}(f_0)\}$$

$$E_2^*(f_0) = h\{(2f_0^2\beta^2 - p^2)(2f_0^2\alpha^2 + p^2)(g_1^0 + g_2^0) - 4f_0^2 g_3^0(p^2 + \alpha^2 f_0^2)\}$$
$$+ (2f_0^2\beta^2 - p^2)\{(f_0^2 - g_1^0 g_2^0)(p^2 + \alpha^2(f_0^2 + g_1^0 g_2^0))$$
$$- \alpha^2 f_0^2 (g_1^0 + g_2^0)^2\} + 4f_0^2\alpha^2 g_1^0 g_2^0 g_3^0 (g_1^0 + g_2^0 - h)$$

$$E_2^{**}(f_0) = h\{(2f_0^2\beta^2 - p^2)(g_1^0 + g_2^0)(2p^2 + 4\alpha^2 f_0^2)$$
$$- (2f_0^2\beta^2 + p^2)(p^2 + \alpha^2 f_0^2 + \alpha^2 g_1^0 g_2^0)\, g_3^0\}$$
$$+ (2f_0^2\beta^2 - p^2)\{p^2 f_0^2 + \alpha^2 f_0^4 - 3\alpha^2 f_0^2 g_1^0 g_2^0$$
$$- 2p^2 g_1^0 g_2^0 - 2\alpha^2(g_1^0 g_2^0)^2\}$$
$$- (2f_0^2\beta^2 - p^2)\{(g_1^0)^2 + (g_2^0)^2\}\,\alpha^2 f_0^2$$
$$+ (2f_0^2\beta^2 + p^2)(g_1^0 + g_2^0)\,\alpha^2 g_1^0 g_2^0 g_3^0$$

$$E_3(f_0) = (g_1^0 - g_2^0)(h - g_1^0)(p^2 - \alpha^2\zeta_1^2)$$

$$(2f_0^2\beta^2 - p^2)(h - g_2^0)(p^2 - \alpha^2\zeta_2^2)\,E_4(f_0)$$
$$= f_0^2(2f_0^2\beta^2 - p^2)\{(h - g_2^0)(p^2 - \alpha^2\zeta_2^2) - (h - g_1^0)(p^2 - \alpha^2\zeta_1^2)\}$$
$$+ (2f_0^2\beta^2 - p^2)\{(g_1^0)^2 (h - g_2^0)(p^2 - \alpha^2\zeta_2^2)$$
$$- (g_2^0)^2 (h - g_1^0)(p^2 - \alpha^2\zeta_1^2)\}$$
$$- 4f_0^2 g_3^0 \{g_1^0 (h - g_2^0)(p^2 - \alpha^2\zeta_2^2) - g_2^0 (h - g_1^0)(p^2 - \alpha^2\zeta_1^2)\}$$

and wherein any superscript zero indicates that such an expression is evaluated at $f = f_0$ the root of $\triangle(f) = 0$.

Equations (14), (15) and (16) completely characterize the correction terms in the displacement components as well as the temperature due to the curvature of the free surface. Moreover, these show that the correction terms are integrals of linear expressions of the corresponding displacement components and temperatures when the free surface is a plane. Even though the expressions appear cumbersome these may be evaluated by the method of steepest descents for large values of p and at large distances from the line sources yielding approximate values required in any numerical calculation. In conclusion, we remark that the Rayleigh wave components may be analyzed similarly.

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