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A STUDY OF VISCOUS FLOW BETWEEN POROUS DISCS THROUGH PARAMETER DIFFERENTIATION

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A general non-iterative method of parameter differentiation is used to obtain the solution of the nonlinear boundary value problem of viscous flow between two porous discs. The problem is first converted into an initial value problem with a parameter, here the injection Reynolds number, as the independent variable and thus the differentiation with respect to the original variable is eliminated. Starting from known value of the flow function for some initial value of the parameter, the solution is extended to a wide range of the later through step by step integration.

INTRODUCTION

The problem of fluid flow between parallel discs, whether fixed or rotating, has been of considerable interest due to its application in the design of thrust bearings, radial diffusers etc. Flow through conduits with porous walls and uniform suction or injection has been previously examined by Berman (1953), Batchelor (1951), Sellars (1955), Yuan (1956) and Yuan and Finkelstein (1956). Savage (1964) analysed the radial flow between parallel plates by perturbing the creeping flow solution. Elkouh (1967) reconsidered the problem of flow between two discs and found a series solution, valid for small values of injection and suction Reynolds number. Rudraiah and Chandrashekhara (1969a, b) made its extension for a conducting fluid for high suction and injection Reynolds number.

The purpose of this paper is to solve the problem considered by Elkouh (1967) through one parameter differentiation technique, as explained by Morel *et al.* (1976). The nonlinear ordinary differential equation governing the flow phenomenon has been converted into a set of first order simultaneous differential equations. The two point boundary value problem is then expressed as an initial value problem which is integrated along a path in the parameter space to obtain the solution for some values of the parameter noniteratively. The method may be extended to a large range of the parameter in a similar fashion. The method thus provides a good tool to study a phenomenon widely relative to the governing parameters, as is often desired in many engineering applications.

BASIC EQUATIONS

Consider the steady axisymmetric flow of a viscous incompressible fluid between two porous discs coinciding with the plane $z = -h$ and $z = h$, as shown in Fig. 1.

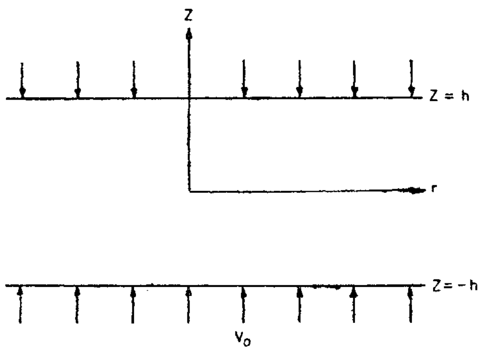

FIG. 1.

Let u, v, w be the velocity components in cylindrical coordinate system. The equations of motion and continuity are:

$$u\frac{\partial u}{\partial r} + \frac{w}{h}\frac{\partial u}{\partial \xi} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{1}{h^2}\frac{\partial^2 u}{\partial \xi^2}\right) \quad \ldots(1)$$

$$u\frac{\partial w}{\partial r} + \frac{w}{h}\frac{\partial w}{\partial \xi} = -\frac{1}{h}\frac{\partial p}{\partial \xi} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{h^2}\frac{\partial^2 w}{\partial \xi^2}\right) \quad \ldots(2)$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{h}\frac{\partial u}{\partial \xi} = 0 \quad \ldots(3)$$

where $\xi = z/h$, p is the pressure and ρ the density.

The boundary conditions are the no slip conditions, viz.

$$\left.\begin{aligned} u(r, \pm 1) &= 0 \\ w(r, \pm 1) &= \mp v_0 = \text{constant} \end{aligned}\right\} \quad \ldots(4)$$

where v_0 is the constant injection velocity. Since the flow is symmetric about the plane $z = 0$, the normal component of velocity and the shear are zero, i.e.

$$w = 0, \quad \frac{\partial u}{\partial z} = 0 \text{ at } z = 0. \quad \ldots(5)$$

Following Elkouh (1967), we define a stream function,

$$\psi(r, \xi) = \frac{v_0 r^2}{2} f(\xi)$$

so that

$$u = \frac{1}{rh}\frac{\partial \psi}{\partial \xi} = \frac{v_0 r}{2h} f'(\xi) \quad \ldots(6)$$

and

$$w = -\frac{1}{r}\frac{\partial \psi}{\partial r} = -v_0 f(\xi). \qquad \text{...(7)}$$

Substituting for u and w in (1) and (2) we get

$$\frac{v_0 r}{2h}\left(\frac{v_0}{2h} f'^2 - \frac{v_0}{h} ff'' - \frac{\nu}{h^2} f'''\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \text{...(8)}$$

$$\frac{v_0^2}{h} ff' + \nu \frac{v_0}{h^2} f'' = -\frac{1}{\rho}\frac{1}{h}\frac{\partial p}{\partial \xi}. \qquad \text{...(9)}$$

Eliminating p from (8) and (9) and integrating, we obtain

$$f''' + R(ff'' - \tfrac{1}{2} f'^2) = C \qquad \text{...(10)}$$

where $R(= v_0 h/\nu)$ is the injection Reynolds number and C is a constant of integration to be determined.

The boundary conditions reduce to

$$\left.\begin{aligned} f'(\pm 1) &= 0 \\ f(\pm 1) &= \pm 1. \end{aligned}\right\} \qquad \text{...(11)}$$

SOLUTION

The nonlinear differential equation (10) can be written as a set of simultaneous differential equations,

$$y_1' = y_2, \quad y_2' = y_3, \quad y_3' = C - R(y_1 y_3 - \tfrac{1}{2} y_2^2) \qquad \text{...(12)}$$

where y_1 is identified with f.

Due to symmetry, the flow is considered in the upper half region $0 \leqslant \xi \leqslant 1$. Correspondingly the boundary conditions are

$$\left.\begin{aligned} y_1(0) = 0, \;\; y_1(1) = 1 \\ y_3(0) = 0, \;\; y_2(1) = 0. \end{aligned}\right\} \qquad \text{...(13)}$$

We now explain the method adopted here. However the details of the method can be seen from Morel *et al.* (1976). The basic theory behind this method is that the given boundary value problem is converted to an initial value problem in which the original independent variable is replaced by the parameter involved in the equation. Therefore finally the equations are discretized with respect to the parameter only, thus getting the results for a wide range of the parameter.

Writing the system of eqns. (12) together with the boundary conditions (13) in the general form as

$$\left.\begin{aligned} &\frac{dy_i}{d\xi} = f_i(y_k, \xi, c, \lambda), \quad i, k = 1, 2, 3 \\ &B_s(y_k(0),\ y_l(1), \lambda, c) = 0, \quad s = 1, 2, 3, 4 \end{aligned}\right\} \qquad ...(14)$$

where $k, l = 1, 2, 3$ and λ is a parameter later to be replaced by the injection Reynolds number R. We further make the following notations :

$$\left.\begin{aligned} &y_i'(\xi) = \frac{\partial y_i}{\partial \xi}, \quad y_{i\lambda} = \frac{\partial y_i}{\partial \lambda}, \quad c_\lambda = \frac{dc}{d\lambda} \\ &y_i(0, \lambda) = a_i(\lambda), \quad y_j(1, \lambda) = b_i(\lambda) \end{aligned}\right\} \qquad ...(15a)$$

and

$$\left.\begin{aligned} &A_{ij} = \frac{\partial f_i}{\partial y_j}, \quad S_i = \frac{\partial f_i}{\partial c}; \ D_i = \frac{\partial f_i}{\partial} \\ &\alpha_i = \frac{da_i}{d\lambda}, \quad \beta_i = \frac{db_i}{d\lambda}. \end{aligned}\right\} \qquad ...(15b)$$

As worked out by Morel *et al.* (1976), making some simplifications and using the method of variation of constants, one can obtain

$$\begin{aligned} y_{i\lambda} = \phi_{ij}\alpha_j + c_\lambda \int_0^\xi \phi_{ik}(\xi, \lambda)\, \phi_{kj}^{-1}(\eta, \lambda)\, S_j(\eta, \lambda)\, d\eta \\ + \int_0^\xi \phi_{ik}(\xi, \lambda)\, \phi_{kj}^{-1}(\eta, \lambda)\, D_j(\eta, \lambda)\, d\eta \end{aligned} \qquad ...(16a)$$

$$\begin{aligned} \beta_i(\lambda) = \phi_{ij}(1, \lambda)\, \alpha_j + c_\lambda \int_0^1 \phi_{ik}(1, \lambda)\, \phi_{kj}^{-1}(\xi, \lambda)\, S_j(\xi, \lambda)\, d\xi \\ + \int_0^1 \phi_{ik}(1, \lambda)\, \phi_{kj}^{-1}(\xi, \lambda)\, D_j(\xi, \lambda)\, d\xi \end{aligned} \qquad ...(16b)$$

where ϕ_{kj}^{-1} are the components of the matrix inverse of ϕ_{ij} which itself is the solution of the fundamental matrix equation

$$\phi_{ij}' = A_{ik}\phi_{kj}; \ \phi_{ij}(0, \lambda) = \delta_{ij} \qquad ...(17)$$

(dash denotes differentiation with respect to ξ). This being treated as a set of ordinary differential equations is solved for the initial value of the parameter by any method. Now differentiating (17) with respect to λ,

$$\left.\begin{aligned} &\phi_{ij\lambda}' = A_{ik}\phi_{kj\lambda} + \phi_{kj}\frac{d}{d\lambda}(A_{ij}) \\ &\phi_{ij\lambda}(0, \lambda) = 0 \\ &\text{and} \\ &\frac{d}{d\lambda}A_{ij} = \frac{\partial A_{ij}}{\partial y_l}y_{l\lambda} + \frac{\partial A_{ij}}{\partial c}c_\lambda + \frac{\partial A_{ij}}{\partial \lambda}. \end{aligned}\right\} \qquad ...(18a)$$

Then applying variation of constants to (18*a*) we obtain

$$\phi_{ij\lambda}(\xi, \lambda) = \int_0^{\xi} \phi_{ik}(\xi, \lambda)\, \phi_{kl}^{-1}(\eta, \lambda)\, \phi_{pj}(\eta, \lambda)\, \frac{dA_{lp}(\eta, \lambda)}{d\lambda}\, d\eta \qquad \ldots(18b)$$

The algorithm of the method is as follows :

(i) The interval [0, 1] is discretized into twenty sub-intervals by taking $h = 0.05$. The value of y_i is then taken from the analytical solution obtained by Elkouh (1967) at each node of the interval for the initial value of the parameter $\lambda = \lambda_0$, while ϕ_{ij} is obtained by solving eqn. (17) at $\lambda = 1$. Here $\lambda_0 = 1$.

(ii) Using (16*b*), the values of the various constants namely β_i and C are obtained from Simpson's rule.

(iii) Having the values of y_i and ϕ_{ij} at the initial value of R and each node ξ, eqns. (16*a*) and (18*b*) give a set of linear ordinary differential equation with independent variable as λ when the integrals are evaluated with the help of Simpsons and Trapezoidal rule. This set of differential equations in λ are then solved by Adam Bashforth predictor corrector formula to obtain y_i and ϕ_{ij} successively for different values of the parameter λ from λ_0 onwards.

Now turning back to the problem, the various matrix functions defined in (15) reduce to

$$A_{ij} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -Ry_3 & -Ry_1 & Ry_2 \end{bmatrix} \qquad \ldots(19)$$

$$S_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad D_i = \begin{bmatrix} 0 \\ 0 \\ -(y_1 y_3 - \frac{1}{2} y_2^2)\, R \end{bmatrix} \qquad \ldots(20)$$

where R_λ denotes differentiation of R with respect to λ. The relation (15*b*) gives

$$\alpha_1 = 0, \ \alpha_3 = 0, \ \beta_1 = 0, \ \beta_2 = 0. \qquad \ldots(21)$$

Using relation (16*b*). we ultimately have the following equations yielding three unknown α_2, β_3 and C_λ,

$$0 = \phi_{12}(1, \lambda)\, \alpha_2 + C \int_0^1 \phi_{1k}(1, \lambda)\, \phi_{k3}^{-1}(\eta, \lambda)\, d\eta$$

$$+ \int_0^1 \phi_{1k}(1, \lambda)\, \phi_{k3}^{-1}(\eta, \lambda)\, D_3(\eta, \lambda)\, d\eta \qquad \ldots(22)$$

$$0 = \phi_{22}(1, \lambda)\, \alpha_2 + C\lambda \int_0^1 \phi_{2k}(1, \lambda)\, \phi_{k3}^{-1}(\eta, \lambda)\, d\eta$$

$$+ \int_0^1 \phi_{2k}(1, \lambda)\, \phi_{k3}^{-1}(\eta, \lambda)\, D_3(\eta, \lambda)\, d\eta \qquad \ldots(23)$$

$$\beta_3 = \phi_{32}(1, \lambda)\, \alpha_2 + C\lambda \int_0^1 \phi_{3k}(1, \lambda)\, \phi_{k3}^{-1}(\eta, \lambda)\, d\eta$$

$$+ \int_0^1 \phi_{3k}(1, \lambda)\, \phi_{k3}^{-1}(\eta, \lambda)\, D_3(\eta, \lambda)\, d\eta. \qquad \ldots(24)$$

The solution for the starting value of $\lambda = 1$ is taken from Elkouh (1967) as:

$$y_1 = \frac{\xi}{2}(3 - \xi^2) + \frac{\lambda}{80}\left(\frac{19}{7}\xi - \frac{39}{7}\xi^3 + 3\xi^5 - \frac{\xi^7}{7}\right)$$

$$+ \frac{\lambda^2}{280}\left(-\frac{137}{385}\xi - \frac{403}{1808}\xi^3 + 1.7\xi^5 - \frac{177}{140}\xi^7 + \frac{1}{6}\xi^9 - \frac{3}{440}\xi^{''}\right). \qquad \ldots(25)$$

Now making use of step (iii) of the algorithm, the values of ϕ_{ij} and y_i are obtained for $R = 1.2, 1.4, 1.6, 1.8$ and 2.0, 10 and -10.

Further defining a pressure coefficient

$$p^* = \frac{p(r, \xi) - p(a, 1)}{\frac{1}{2}\rho v_0^2}\left(\frac{h}{a}\right)^2. \qquad \ldots(26)$$

We obtain the pressure distribution on the disc as

$$p^* = \frac{1}{2R}(-C)\left(1 - \frac{r^2}{a^2}\right) \qquad \ldots(27)$$

where a is the radius of the disc. Also the constant C is the same as used in eqn. (10) and its value has been found to be -4.277135.

The shear stress on the disc, is defined as

$$\tau_0 = -\frac{\mu}{h}\left(\frac{\partial u}{\partial \xi}\right)_{\xi=1}. \qquad \ldots(28)$$

The dimensionless skin friction coefficient is given by

$$\tau_0^* = \frac{\tau_0}{\frac{1}{2}\rho v_0^2}\left(\frac{h}{a}\right)$$

$$= -\frac{1}{R}\left(\frac{r}{a}\right) f''(1). \qquad \ldots(29)$$

For fixed $r/a(= 1)$, the values of τ_0^* for different R are given in Table I.

TABLE I

R	1.2	1.4	1.6	1.8	2.0
τ_0^*	2.1367913	2.2053813	2.2629312	2.3121651	2.4327359

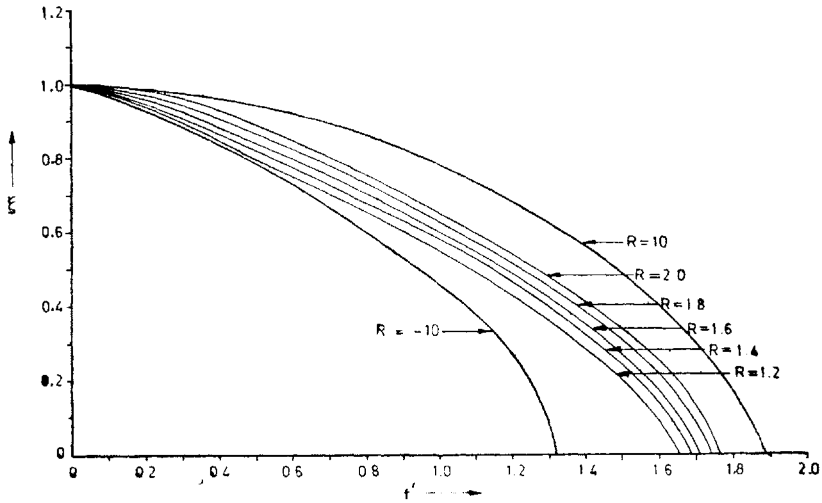

FIG. 2. Radial velocity profile for different R.

CONCLUSIONS

The values of f' (radial velocity) have been plotted for $R = 1.2, 1.4, 1.6, 1.8, 2.0, 10, -10$ (Fig. 2). The results so calculated for low Reynolds number are compared with the results given in Elkouh (1967) which are in good agreement while those for high suction and injection are compared with Rudraiah and Chandrashekhara (1969a). As is expected, the radial velocity is found to increase throughout with increase in Reynolds number. For positive R, the shape is quite parabolic more pronounced for high R. For negative R, the curve flattens a bit, more for high (numerically) negative R. Also from Table I, it is evident that skin friction is more for higher rates of injection. The results can be extended to any value of R.

Convergency and Stability

Since the method is non-iterative, hence it is quite convergent. Moreover neither in the derivation of the method, nor in its application, nowhere big approximations are used. The approximations that are used, are in finding (i) y_i and ϕ_{ij} at initial value of R; (ii) various integrals by Trapezoidal and Simpson's rule; (iii) y_i and ϕ_{ij} back from $y_{i\lambda}$ and $\phi_{ij\lambda}$ by Adam Bashforth formula. But all of them are highly accurate (error of $O(h^4)$). So the method is quite stable.

The method is most appropriate in the problems in which a parameter occurs naturally. It is also used in some problems, which are difficult to solve as it is, by introducing an artificial parameter. However the large number of equations may prove to be a drawback, but since the derivatives of the unkowns are expressed explicitely in terms of unknowns, most suitable to digital computation, manipulations are faster.

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