Indian J. pure appl. Math., **11**(9) : 1225–1234, September 1980

HYDROMAGNETIC EKMAN LAYER OVER AN OSCILLATORY POROUS PLATE

D. V. KRISHNA AND D. R. V. PRASADA RAO

Department of Mathematics, Post-Graduate Centre, Anantapur 515003 (*A.P.*)

(*Received* 31 *December* 1979)

The Hall effects on the Ekman layer in an electrically conducting rotating fluid bounded by an infinite oscillatory porous plate under a uniform transverse magnetic field are considered. The expressions for velocity, temperature in the boundary layer and the skin friction on the plate are derived and discussed numerically. The effects resulting from the interaction of the force induced by the Hall current with the Coriolis and Lorentz forces are clearly brought out in the various profiles of the velocity and temperature distribution.

1. INTRODUCTION

The MHD flow of a viscous, incompressible and electrically conducting fluid in a parallel plate channel caused by impulsive or oscillatory movements of one or both the boundaries has been extensively studied by several authors under varied conditions. When the strength of the magnetic field is very large, the Hall effects play a significant role in determining the flow features. Sato (1961), Yamanishi (1962), Sherman and Sutton (1961) have discussed the Hall effects on the steady hydromagnetic flow between two parallel plates. These effects in the unsteady case were discussed by Vatazhin (1965), Pop (1971, 75), Sakhnovskii (1963). Taking the infinite plate to be porous, Gupta (1975) has studied the Hall effects in the steady hydromagnetic flow of an electrically conducting fluid. This analysis is extended to include the free convection effects in the vertical plate configuration by Datta and Mazumdar (1976). Datta and Jana (1976) have considered the Hall effects on the MHD flow past a flat plate in the presence of a uniform transverse magnetic field when the free stream oscillates in magnitude. It has been found that for small frequency of oscillations of the free-stream the increase in Hall parameter increases the phase difference of the oscillations of the shearing stress. And also the resonance frequency of the boundary layer decreases with the increase in Hall parameter. Mazumdar *et al.* (1976) have extended the analysis of Gupta (1975) to a rotating frame by considering the hydromagnetic Ekman layer in a semi-infinite rotating fluid bounded by a porous plate maintained at a higher temperature than that of the free stream. Neglecting the ion slip, the thermoelectric effect, the electronic pressure and assuming that the free-stream moves with a uniform velocity, it has been observed that the Ekman layer thickness increases with increase in the

Hall parameter for fixed magnetic parameter but decreases with increase in the strength of the magnetic field.

In this paper we consider the Hall effects on the Ekman layer in an electrically conducting rotating fluid bounded by an infinite oscillatory porous plate under a uniform transverse magnetic field. The expressions for velocity, temperature in the boundary layer and the skin friction are derived and discussed numerically. The effects resulting from the interaction of the force induced by the Hall current with the Coriolis and Lorentz force are clearly brought out in the various profiles corresponding to the velocity and temperature distributions.

2. Governing Equations and Solution

Consider the oscillating flow of an electrically conducting fluid bounded by an infinite porous flat plate $z' = 0$. The whole system is rotating with a constant angular velocity about the z-axis. A uniform transverse magnetic field is acting parallel to the axis of rotation. Taking the magnetic Reynolds number to be small, the induced magnetic field is neglected in comparison with the applied magnetic field. Since the plate is infinite in extent, all the physical variables except pressure depend on z' and t' only. In view of the continuity equation, it follows that $w' = w'(t')$ and we assume

$$w'(t') = -\, w_0(1 + \epsilon A \exp(i\omega' t')$$

where ω' is the frequency, A is a real positive constant, ϵ is a small parameter such that $\epsilon A \ll 1$, w_0 is a nonzero positive constant mean suction velocity. The conservation of electric charge $\nabla \cdot \mathbf{J} = 0$ gives $J_z =$ constant, where $\mathbf{J} = (J_x, J_y, J_z)$. This constant is zero since $J_z = 0$ at the plate which is electrically non-conducting. Thus $J_z = 0$ everywhere in the flow.

When the strength of the magnetic field is very large, Ohm's law must be modified, to include Hall currents as follows

$$\mathbf{J} + \frac{\omega_e \tau_e}{H_0}(\mathbf{J} \times \mathbf{H}) = \sigma\left(\mathbf{E} + \mu_e \mathbf{q} \times \mathbf{H} + \frac{1}{e\eta_e}\nabla p_e\right) \qquad \ldots(2.1)$$

where E is the electric field, ω_e is the cyclotron frequency and τ_e is the collision time of electrons, e is the electric charge, η_e is number density of electrons and p_e the electron pressure. In writing (2.1), the ionslip and the thermo-electric effects are neglected and further it is assumed that $\omega_e \tau_e \ll 1$. For partially ionised gases, the electron pressure gradient may be neglected (Sato 1961), then eqn. (2.1) gives

$$J_x - \omega_e \tau_e J_y = \sigma(E_x - \mu_e H_0 w') \qquad \ldots(2.2)$$

$$J_y + \omega_e \tau_e J_x = \sigma(E_y + \mu_e H_0 u') \qquad \ldots(2.3)$$

where μ_e is the magnetic permeability and σ is the electrical conductivity.

In the free-stream, the magnetic field is uniform and $\nabla \times \mathbf{H} = \mathbf{J}$ shows that there is no electric current. Thus

$$J_x \to 0 \text{ and } J_y \to 0 \text{ as } z' \to \infty. \qquad \ldots(2.4)$$

Further it is assumed that

$$u' = U'(t') \text{ and } v' \to 0 \text{ as } z' \to \infty \qquad \ldots(2.5)$$

where $U'(t') = U_0(1 + \epsilon \exp(i\omega' t'))$ is the free stream velocity. Using (2.4) and (2.5) in (2.2) and (2.3), we obtain

$$E_x = 0,\ E_y = \mu_e H_0 U'(t') \text{ as } z' \to \infty. \qquad \ldots(2.6)$$

Further since $\nabla \times E = -\mu_e \dfrac{\partial \mathbf{H}}{\partial t'} = 0$ as the magnetic field is uniform we obtain

$$\frac{dE_x}{dz'} = 0 \text{ and } \frac{dE_y}{dz'} = 0$$

and eqn. (2.6) then gives

$$E_x = 0 \text{ and } E_y = \mu_e H_0 U'(t') \text{ everywhere}. \qquad \ldots(2.7)$$

The unsteady hydromagnetic boundary layer equations with respect to the rectangular coordinate system rotating with the same angular velocity Ω, in component form are

$$\frac{\partial u'}{\partial t'} - w_0(1 + \epsilon A \exp(i\omega' t'))\frac{\partial u'}{\partial z'} - 2\Omega v' = -\frac{1}{\rho}\frac{\partial p'}{\partial x'} + \nu \frac{\partial^2 u'}{\partial z'^2} + \frac{\mu_e H_0}{\rho} J_y \qquad \ldots(2.8)$$

$$\frac{\partial v'}{\partial t'} - w_0(1 + \epsilon A \exp(i\omega' t'))\frac{\partial v'}{\partial z'} + 2\Omega u' = -\frac{1}{\rho}\frac{\partial p'}{\partial y'} + \nu \frac{\partial^2 v'}{\partial z'^2} - \frac{\mu_e H_0}{\rho} J_x \qquad \ldots(2.9)$$

$$0 = -\frac{1}{\rho}\frac{\partial p'}{\partial z'} \qquad \ldots(2.10)$$

$$\rho c_p\left[\frac{\partial T}{\partial t'} - w_0(1 + \epsilon A \exp(i\omega' t'))\frac{\partial T}{\partial z'}\right] = k\frac{\partial^2 T}{\partial z^2} + \mu\left[\left(\frac{\partial u'}{\partial z'}\right)^2 + \left(\frac{\partial v'}{\partial z'}\right)^2\right] \qquad \ldots(2.11)$$

The boundary conditions are

$$u' = 0,\ v' = 0,\ T = T_0 \text{ on } z' = 0 \qquad \ldots(2.12)$$

$$u' \to U'(t'),\ v' \to 0,\ T \to T_\infty \text{ as } z' \to \infty \qquad \ldots(2.13)$$

where ρ is the density, ν the kinematic viscosity, p' the pressure including the centrifugal force, k the coefficient of thermal expansion and c_p the specific heat of the fluid.

Under the usual boundary layer approximation the pressure can be obtained from the free-stream as

$$-\frac{1}{\rho}\frac{\partial p'}{\partial x'} = \frac{\partial U'}{\partial t'} \text{ and } -\frac{1}{\rho}\frac{\partial p'}{\partial y'} = 2\Omega U'(t'). \qquad \text{...(2.14)}$$

Substituting (2.14) eqns. (2.8) and (2.9) reduce to

$$\frac{\partial(u' - U')}{\partial t'} - w_0(1 + \epsilon A \exp(i\omega' t'))\frac{\partial u'}{\partial z'} - 2\Omega v'$$

$$= \nu\frac{\partial^2 u'}{\partial z'^2} + \frac{\mu_e H_0}{\rho} J_y \qquad \text{...(2.15)}$$

$$\frac{\partial v'}{\partial t'} - w_0(1 + \epsilon A \exp(i\omega' t'))\frac{\partial v'}{\partial z'} + 2\Omega(u' - U') = \nu\frac{\partial^2 v'}{\partial z'^2} - \frac{\mu_e H_0}{\rho} J_x. \qquad \text{...(2.16)}$$

We introduce the following non-dimensional variables:

$$z = \frac{|w_0|}{\nu} z', \; t = \frac{w_0^2}{4\nu} t', \; \omega = \frac{4\nu}{w_0^2}\omega', \; (u, v) = U_0(u', v'),$$

$$U = U_0 U', \; \theta = \frac{T - T_0}{T_\infty - T_0}.$$

Eliminating J_x and J_y from eqns. (2.15) – (2.16) using (2.2) – (2.3), the governing equations in the non-dimensional form reduce to

$$\left[\tfrac{1}{4}\frac{\partial}{\partial t} - (1 + \epsilon A \exp(i\omega t))\frac{\partial}{\partial z} - \frac{\partial^2}{\partial z^2}\right] q + (\alpha + i(2E + \alpha m))(q - U)$$

$$= \tfrac{1}{4}\frac{\partial U}{\partial t} \qquad \text{...(2.17)}$$

$$\tfrac{1}{4}\frac{\partial \theta}{\partial t} + (1 + \epsilon A \exp(i\omega t))\frac{\partial \theta}{\partial z} = \frac{1}{P_r}\frac{\partial^2 \theta}{\partial z^2} + E_e\left(\frac{dq}{dz}\cdot\frac{\overline{dq}}{dz}\right) \qquad \text{...(2.18)}$$

where

$$q = u + iv$$

$$E = \frac{\Omega\nu}{|w_0|}, \; M^2 = \frac{\sigma\nu\mu_e^2 H_0}{\rho w_0^2}, \; \alpha = \frac{M^2}{1 + m^2}, \; P_r = \frac{\mu c_p}{k}, \; E_e = \frac{U^2}{c_p(T_0 - T_\infty)}$$

$$m = \omega_e \tau_e.$$

The boundary conditions are

$$\begin{aligned} &q = 0, \theta = 0 \ \text{ at } \ z = 0 \\ &q \to U(t), \theta \to 1 \ \text{ as } \ z \to \infty. \end{aligned} \qquad \text{...(2.19)}$$

Following Lighthill (1954) we assume $q(z, t)$ and $U(t)$ as

$$\begin{aligned} q(z, t) &= [1 - q_0(z)] + [1 - q_1(z)]\, \epsilon \exp(i\omega t) + \dots \\ &\quad + [1 - q_n(z)]\, \epsilon \exp(ni\omega t) \\ U(t) &= 1 + \epsilon \exp(i\omega t) + \epsilon^2 \exp(2i\omega t) + \dots + \epsilon^n \exp(ni\omega t). \end{aligned} \qquad \text{...(2.20)}$$

Substituting (2.20) in (2.17) and equating the like terms on both sides, we get

$$\left.\begin{aligned} &q_0'' + q_0' - (\alpha + i(2E + \alpha m))\, q_0 = 0 \\ &q_1'' + q_1' - \left(\alpha + i\left(2E + \alpha m + \frac{\omega}{4}\right)\right) q_1 = -Aq_0' \\ &q_2'' + q_2' - \left(\alpha + i\left(2E + \alpha m + \frac{\omega}{2}\right)\right) q_2 = -Aq_1' \\ &\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad \vdots \qquad\qquad\quad \vdots \\ &q_n'' + q_n' - \left(\alpha + i\left(2E + \alpha m + \frac{n\omega}{4}\right)\right) q_n = -Aq_{n-1}' \end{aligned}\right\} \qquad \text{...(2.21)}$$

where dash denotes differentiation with respect to z. The reduced boundary conditions are

$$\left.\begin{aligned} &q_0 = q_1 = \dots = q_n = 1 \ \text{ on } \ z = 0 \\ &q_0 = q_1 = \dots = q_n = 0 \ \text{ as } \ z \to \infty \end{aligned}\right\} \qquad \text{...(2.22)}$$

Solving the above equations, we obtain

$$\begin{aligned} q_0(z) &= \exp(-L_0 z) \\ q_1(z) &= B \exp(-L_0 z) + (1 - B) \exp(-L_1 z) \\ q_2(z) &= \exp(-L_2 z) - 8A^2 L_0^2/\omega^2 \times (\exp(-L_0 z) - \exp(-L_2 z)) \\ &\quad + \left(\frac{16A^2 L_0 L_1}{\omega^2} + \frac{4iAL_1}{\omega}\right) (\exp(-L_1 z) - \exp(-L_2 z)) \end{aligned}$$

where

$$\begin{aligned} L_0 &= [1 + ((1 + 4\alpha) + 4i(2E + \alpha m))^{1/2}]^{1/2} \\ L_1 &= [1 + ((1 + 4\alpha) + 4i(2E + \alpha m + \omega/4))^{1/2}]^{1/2} \\ L_2 &= [1 + ((1 + 4\alpha) + 4i(2E + \alpha m + \omega/2))^{1/2}]^{1/2} \\ B &= 4iAL_0/\omega. \end{aligned}$$

Similarly the expressions for q_3, q_4, ... etc., can be obtained.

The velocity distribution $(u + iv)$ of a hydromagnetic flow inside the boundary layer is

$$\begin{aligned} q(z, t) &= u(z, t) + iv(z, t) \\ &= [1 - \exp(-L_0 z)] \\ &\quad + \epsilon \exp(i\omega t)\,[1 - B\exp(-L_0 z) - (1 - B)\exp(-L_1 z)] \\ &\quad + \epsilon^2 \exp(2i\omega t)\Big[\exp(-L_2 z) - \frac{8A^2L_0^2}{\omega^2}(\exp(-L_0 z) - \exp(-L_2 z)) \\ &\quad + \left(\frac{16A^2L_0L_1}{\omega^2} + \frac{4iAL_1}{\omega}\right)(\exp(-L_1 z) - \exp(-L_2 z))\Big]. \qquad \ldots(2.23) \end{aligned}$$

The non-dimensional expression for skin friction τ is

$$\begin{aligned} \tau = \tau_x + i\tau_y &= (\partial q/\partial z)_{z=0} \\ &= L_0 + \epsilon \exp(i\omega t)\,(L_1(1 - B) + BL_0) \\ &\quad + \epsilon^2 \exp(2i\omega t)\Big[L_2 + \left(\frac{8A^2L_0^2(L_2 - L_0)}{\omega^2}\right) \\ &\quad - \left(\frac{16A^2L_0L_1}{\omega^2} + \frac{4iAL_1}{\omega}\right)(L_2 - L_1)\Big] \qquad \ldots(2.24) \end{aligned}$$

The expressions for $u(z, t)$, $v(z, t)$, τ_x and τ_y are obtained by taking the real and imaginary parts and their behaviour for various parametric values is discussed by plotting the corresponding profiles.

Having obtained $q(z, t)$, the temperature distribution $\theta(z, t)$ is solved assuming

$$\begin{aligned} \theta(z, t) &= \theta_0(z) + \epsilon\theta_1(z)\exp(i\omega t) + \epsilon^2\theta_2(z)\exp(2i\omega t) + \ldots \\ &\quad + \epsilon^n\theta_n(z)\exp(ni\omega t). \end{aligned}$$

The behaviour of θ is determined analytically and the profiles for θ are drawn for different parametric values.

3. Discussion of the Results

It can be observed from the expressions for the velocity components in the plane of rotation, that the steady state distributions are in the form of a logarithmic spiral similar to the Ekman velocity spiral for the rotating flow over a disk. Hence for a small magnetic Reynolds number Hall currents contribute to play a significant role in determining the velocities in the plane of rotation. Also the steady state flow corresponding to $\epsilon \to 0$ exhibits a boundary layer growth of thickness L_{0r}^{-1}. This boundary layer thickness decreases with the increase in the magnetic parameter M for fixed Hall parameter while its thickness increases with increase in the Hall parameter for fixed M. Also the thickness of the layer goes on decreasing with increase in rotation. Near the plate $z = 0$, in the steady state u approximately equals $L_{0r}z$ and v is equal to $L_{0i}z$ so that the resultant flow is inclined at an angle $\tan^{-1}\left(\frac{L_{0i}}{L_{0r}}\right)$ to the direction of the free-stream.

The expressions for the velocity, temperature and skin friction are plotted for different values of the magnetic parameter M, the Hall parameter m, the frequency of oscillation ω, the parameter A corresponding to the unsteady part of the suction velocity, the Prandtl number P_r and the Eckert number E_e on an arbitrary plane of rotation, say $z = 1$.

Figures 1 and 2 represent the variation of the non-dimensional velocity components for different values of M and m. The unbroken lines represent the velocity u and the broken lines represent the velocity v.

The profiles for u indicate that its magnitude decreases with the increase in m for fixed M, and increases with the increase in M for fixed m. The y-component of v increases with m for fixed M and decreases with M for fixed m. The change in v is much rapid comparable to the corresponding change in the x-component of the velocity u.

Also the transient and total velocity profiles are plotted in Figs. 3 and 4 for different ω and A. For a fixed A, u goes on decreasing with increase in ω and increases with increase in A for fixed ω. The component v in this case increases with increase in the frequency of oscillation ω and increases with A for fixed ω.

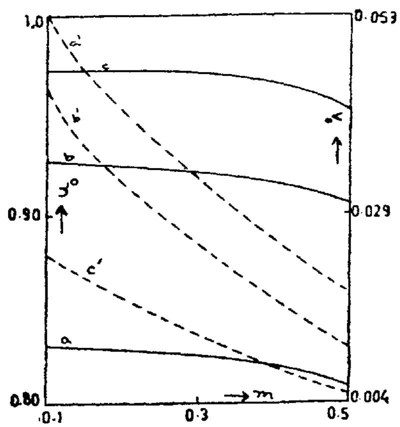

FIG. 1. Steady parts of u and v against m when $\omega t = \pi/2$, $\omega = 1$, $A = 0.2$

	a	b	c	a′	b′	c′
M	1	2	3	1	2	3

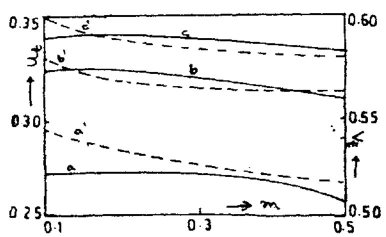

FIG. 2. Transient parts of u and v against an

	a	b	c	a′	b′	c′
M	1	2	3	1	2	3

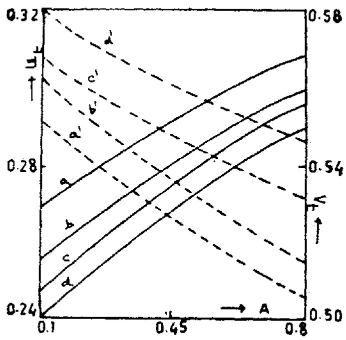

FIG. 3. Transient parts of u and v against A when $M = 1$, $m = 0.5$

	a	b	c	d	a′	b′	c′	d′
ω	1	2	3	4	1	2	3	4

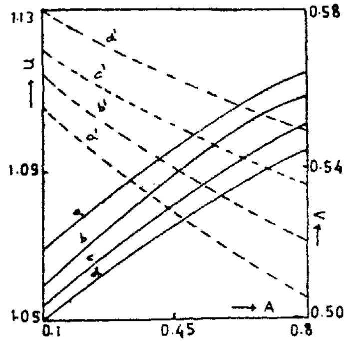

FIG. 4. Total u and v against A

	a	b	c	d	a′	b′	c′	d′
ω	1	2	3	4	1	2	3	4

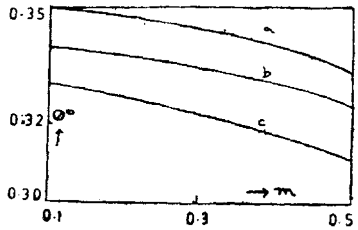

FIG. 5. Steady part of temperature θ against m with $\omega = 1$, $A = 0.2$, $P_r = 1.5$, $E_e = 0.5$

	a	b	c
M	1	2	3

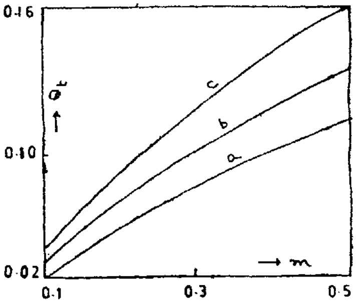

FIG. 6. Transient part of θ against m

	a	b	c
M	1	2	3

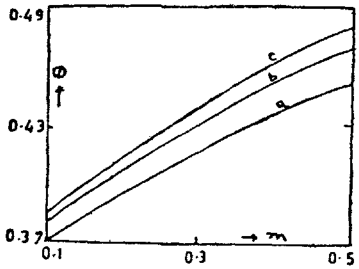

FIG. 7. Total θ against m

	a	b	c
M	1	2	3

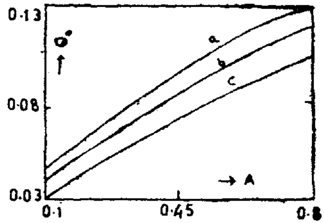

FIG. 8. Steady part of θ against A with $M = 1$, $m = 0.5$, $P_r = 1.5$, $E_e = 0.5$

	a	b	c
ω	1	2	3

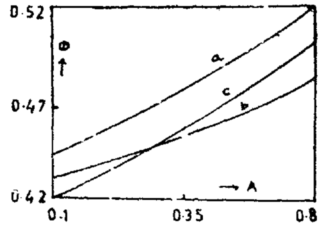

FIG. 9. Total θ against A

	a	b	c
ω	1	2	3

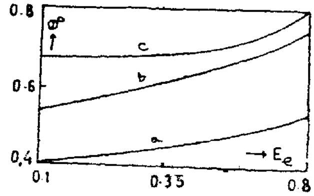

FIG. 10. Steady part of θ against E_e with $\omega = 1, M = 1, A = 0.2, m = 0.5$

	a	b	c
P_r	0.5	1.0	1.5

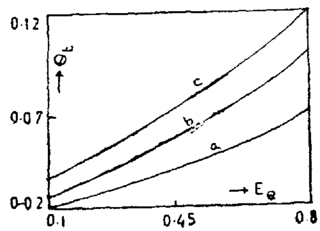

FIG. 11. Transient part of θ against E_e

	a	b	c
P_r	0.5	1.0	1.5

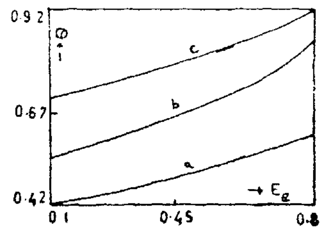

FIG. 12. Total θ against E_e

	a	b	c
P_r	0.5	1.0	1.5

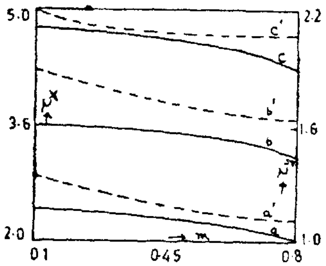

FIG. 13. τ_x and τ_y against m when $\omega = 1, A = 0.2$

	a	b	c	a′	b′	c′
M	1	2	3	1	2	3

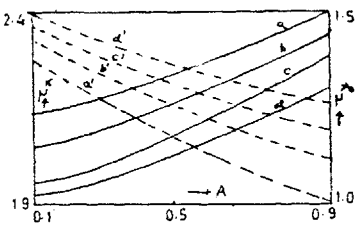

FIG. 14. τ_x and τ_y against A with $M = 1, m = 0.5$

	a	b	c	d	a′	b′	c′	d′
ω	1	2	3	4	1	2	3	4

The steady, transient and total temperature profiles are plotted in Figs. 5, 6, 7 respectively for different M and m. It is observed that the steady temperature monotonically decreases in a slowpace with either increase in m for M or vice-versa. The nature of transient and total temperature profiles is different from that of steady temperature. Both the transient and total temperature increase rapidly with increase in m for fixed M and vice-versa.

Figures 8 and 9 show the variation of transient and total temperature for different values of ω and A. These profiles show that the transient and total temperatures increase monotonically with increase in A for fixed ω but decreases with increase in ω for fixed A.

In Figs. 10, 11 and 12 the variations of steady, transient and total temperatures have been plotted respectively for different values of P_r and E_e. It is inferred from the above figures that they increase with either increase in the Eckert number or the Prandtl number keeping the other fixed.

The x and y components of skin friction are plotted in Fig. 13 for different M and m. The magnetic parameter M increases the x and y components of skin friction for fixed m, but the Hall parameter m decreases the x components and increases the y components of skin friction for fixed M.

The x and y components of the skin friction are plotted for different A and M in Fig. 14. It is evident from the figure that the frequency of oscillation decreases the x-component and increases the y-component of skin friction for fixed A where as both the components increase rapidly with increase in A for fixed value of the frequency of oscillation.

REFERENCES

Cowling, T. G. (1957). Magnetohydrodynamics, Inter-Sciences Publishers, Inc., New York, p. 101.

Datta, N., and Jana, R. N. (1976). Oscillatory magnetohydrodynamic flow past a flat plate with Hall effects. *J. Phys. Soc. Japan.*, **40**, 1469.

Datta, N., and Mazumdar, M. (1976). Hall effects on hydrodynamic free convection past an infinite porous plate. *J. Math. Phys. Sci.*, **10**(1), 59.

Gupta, A. S. (1975). Hydromagnetic flow past a porous flat plate with Hall effects. *Acta Mech.*, **22**, 281.

Lighthill, M. J. (1954). The response of the laminar skin friction and heat transfer to fluctuations in the stream velocity. *Proc. R. Soc.*, **A224**.

Mazumdar, M., Gupta, A. S., and Datta, N. (1976). Flow and heat transfer in the hydromagnetic Ekman layer over a porous plate with Hall effects. *Int. J. Heat Mass Transfer*, **19**, 523.

Pop, I. (1971). The effect of Hall currents on hydromagnetic flow near an accelerated plate. *J. Math. Phy. Sci.*, **5**(4), 375.

——————— (1975). Der Einflu B Von Hall-Straman auf die Rayleigh-Grenzchicht. *ZAMM*, **55**, 149.

Sato, H. (1961). *J. Phys. Soc. Japan*, **16**, 1427.

Sherman, A., and Sutton, G. W. (1961). Magnetohydrodynamics. Evanston, Illinois, p. 173.

Sakhnovskii, E. G. (1963). *Soviet Physics Technical Physics*, **8**, 469.

Vatazhin, A. B. (1965). *Z. Prikl. Mech. Tech. Fiz.*, **2**, 3.

Yamanishi, T. (1962). Preprint, 17*th Annual Meeting*, *Phy. Soc. Japan*, *Osaka*, **5**, 29.