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ELASTIC-PLASTIC BENDING OF WIDE PLATES

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Elastic-plastic transition in a rectangular plate bent into the form of a circular cylinder has been discussed without making a number of adhoc assumptions. The extension and contraction regions are given by the transition points of the differential equation defining the deformed field. The analysis also gives the neutral surface separating the two regions. Numerical results are shown graphically.

1. INTRODUCTION

Elastic-plastic bending of a wide plate has been discussed by Shaffer and House (1955) by making the following assumptions :

(1) It is bent under the condition of plane strain.

(2) In the region of extension $\tau_{\theta\theta} > \tau_{rr}$ $(a \leqslant r \leqslant c)$ and in region of compression $\tau_{rr} > \tau_{\theta\theta}$ $(c \leqslant r \leqslant b)$, where a and b are the radii of the cylindrical boundaries of the deformed state and c is the radius of the neutral surface.

(3) The conditions of yielding

$$\tau_{\theta\theta} - \tau_{rr} = 2k, \ (a \leqslant r \leqslant c)$$

$$\tau_{rr} - \tau_{\theta\theta} = 2k, \ (c \leqslant r \leqslant b) \qquad \text{occurs.}$$

Making use of Seth's (1963b) transition theory it has been shown, in this paper, that there is no need to assume conditions (1) to (3) but they follow from the analysis themselves. Seth's transition theory utilizes the concept of asymptotic solution through the critical points of the differential system defining the deformed field. The state of complete plasticity from the transition results are obtained by making the poisson's ratio $\sigma \rightarrow \frac{1}{2}$.

2. GOVERNING EQUATIONS

We consider an initially wide plate bent into the form of a circular cylinder with two edges as generators. The bending moment M per unit length is applied perpendicular to the plane of the paper. Let (x', y', z') be the coordinates of a point P of the plate before deformation. We assume that the two faces of the plate get bent into right cylindrical surfaces of inner radius a and the outer radius b and the other two into axial terminating planes given by $\theta = \pm z$. The axis of z' is parallel to the axis of the cylinder which we take the axis of z. From the symmetrical bending of the plate it appears that a cross-section perpendicular to the axis of z' remains plane after deformation.

Under these considerations, the displacements are given by Seth (1935, 1962) :

$$\left.\begin{aligned} u &= x - f(r), \quad v = y - A\theta \\ w &= \alpha z \end{aligned}\right\} \qquad \text{...(2.1)}$$

where A and α are constants and $f(r)$ is a function of $r = (x^2 + y^2)^{1/2}$ only.

The finite components of strain referred to the strained states are

$$\left.\begin{aligned} e_{rr} &= \tfrac{1}{2}\left[1 - f'^2\right] \\ e_{\theta\theta} &= \tfrac{1}{2}\left[1 - \left(\frac{A}{r}\right)^2\right] \\ e_{zz} &= \tfrac{1}{2}\left[1 - (1-\alpha)^2\right] \\ e_{r\theta} &= e_{\theta z} = e_{zr} = 0. \end{aligned}\right\} \qquad \text{...(2.2)}$$

Using the stress-strain relations (Sokolinkoff 1956)

$$\tau_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij} \quad (i, j, k = 1, 2, 3) \qquad \text{...(2.3)}$$

where λ and μ are Lames' constants, we obtain for the problem at hand

$$\left.\begin{aligned} \tau_{rr} &= \frac{\lambda + 2\mu}{2}\left[1 - f'^2\right] + \frac{\lambda}{2}\left[1 - \left(\frac{A}{r}\right)^2\right] + \lambda K \\ \tau_{\theta\theta} &= \frac{\lambda + 2\mu}{2}\left[1 - \left(\frac{A}{r}\right)^2\right] + \frac{\lambda}{2}\left[1 - f'^2\right] + \lambda K \\ \tau_{zz} &= (\lambda + 2\mu)K + \frac{\lambda}{2}\left[1 - f'^2\right] + \left[1 - \left(\frac{A}{r}\right)^2\right] \\ \tau_{r\theta} &= 0, \ \tau_{\theta z} = 0, \ \tau_{zr} = 0. \end{aligned}\right\} \qquad \text{...(2.4)}$$

where

$$K = \tfrac{1}{2}\left[1 - (1-\alpha)^2\right].$$

Equations of equilibrium are all satisfied except

$$\frac{\partial\tau_{rr}}{\partial r} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0. \qquad \text{...(2.5)}$$

Substituting the values of τ_{rr}, $\tau_{\theta\theta}$ from eqns. (2.4) in eqn. (2.5) we get a nonlinear differential equation in f:

$$rf'f'' - (1-c)\left(\frac{A}{r}\right)^2 - \frac{c}{2}\left[(1 - f'^2) - \left\{1 - \left(\frac{A}{r}\right)^2\right\}\right] = 0. \qquad \text{...(2.6)}$$

where

$$c = \frac{2\mu}{\lambda + 2\mu} = \frac{1 - 2\sigma}{1 - \sigma}; \quad \sigma \text{ is Poisson's ratio.}$$

Putting $rf' = fP$ in this equation we obtain

$$\frac{df}{dP} = \frac{2f(fP)^2}{2(1-c)\,A^2 + cr^2\left[\left\{1 - \left(\frac{fP}{r}\right)^2\right\} - \left\{1 - \left(\frac{A}{r}\right)^2\right\}\right] - 2(P-1)\,(fP)^2}. \qquad \text{...(2.7)}$$

The transition points of this equation are $P = 0$ and $P = \infty$.

The boundary conditions are

$$\tau_{rr} = 0 \ \text{ at } \ r = a \ \text{ and } \ r = b. \qquad \text{...(2.8)}$$

Over the plane ends, we require

$$\int_a^b r\tau_{zz}\, dr = 0 \qquad \text{...(2.9)}$$

where as on the straight edges $\theta = \pm z$ we have,

$$\int_a^b \tau_{\theta\theta}\, dr = 0 \ \text{ and } \ M = -\int_a^b r\tau_{\theta\theta}\, dr. \qquad \text{...(2.10)}$$

3. Transition and Plastic Stress Components

For finding the plastic stress, the transition function is taken through the principal stress (Seth 1963a, Gupta *et al.* 1979) at the transition point. We take the transition function R as,

$$R = 1 + (1 - c)\,(1 + 2K) - \frac{c}{\mu}\,\tau_{rr} \equiv f'^2 + (1 - c)\left(\frac{A}{r}\right)^2. \qquad \text{...(3.1)}$$

Taking logarithmic differentiation of eqn. (3.1) with respect to r and using eqn. (2.7), we get

$$\frac{d}{dr}\,[\log R] = \frac{\frac{c}{r}\left[\left(\frac{A}{r}\right)^2 - \left(\frac{fP}{r}\right)^2\right]}{\left(\frac{fP}{r}\right)^2 + (1 - c)\left(\frac{A}{r}\right)^2}. \qquad \text{...(3.2)}$$

As $P \to \infty$ which corresponds to an infinite tension, eqn. (3.2) gives

$$\frac{d}{dr}\,[\log R] = -\frac{c}{r}. \qquad \text{...(3.3)}$$

Integrating eqn. (3.3), we have

$$R = A_0 r^c \qquad \text{...(3.)}$$

where A_0 is constant of integration. Similarly, when $P \to 0$ which corresponds to an infinite compression, we obtain from eqn. (3.2) the transition value of R as

$$R = A_1 r^{c/(1-c)}, \qquad \ldots(3.5)$$

where A_1 is the integration constant.

Substituting the values of R from eqns. (3.4) and (3.5) in eqn. (3.1), we obtain the stress τ_{rr} for the region of tension as

$$\tau_{rr} = \frac{\mu}{c}\left[1 + (1 - c)(1 + 2k) - A_0 r^{-c}\right] \qquad \ldots(3.6)$$

and the stress τ_{rr} for the region of compression as

$$\tau_{rr} = \frac{\mu}{c}\left[1 + (1 + c)(1 + 2k) - A_1 r^{c/(1-c)}\right]. \qquad \ldots(3.7)$$

Using the relation $e_{\theta\theta} = \frac{1}{2}[1 - (A/r)^2]$, which shows that $r = A$ is the unstreched longitudinal fibre. If τ^*_{rr} denotes the radial stress τ_{rr} at $r = A$ (the neutral axis) the eqns. (3.6) and (3.7) can be written as,

$$\tau_{rr} = \left(\frac{A}{r}\right)^c \cdot \tau^*_{rr} + \frac{\mu}{c}\left[1 + (1 - c)(1 + 2k)\right]\left[1 - \left(\frac{A}{r}\right)^c\right] \qquad \ldots(3.8)$$

in the region of tension i.e., $a \leqslant r \leqslant A$, and

$$\tau_{rr} = \left(\frac{r}{A}\right)^{c/(1-c)} \cdot \tau^*_{rr} + \frac{\mu}{c}\left[1 + (1 - c)(1 + 2k)\right]\left[1 - \left(\frac{r}{A}\right)^{c/(1-c)}\right] \qquad \ldots(3.9)$$

in the region of compression i.e., $A \leqslant r \leqslant b$. It follows from the results for simple shear under condition of finite deformation (Seth 1963b) that in the transition $\mu \to k$ the latter being the yield limit in shear.

Using the boundary conditions (2.8) in eqns. (3.8) and (3.9) respectively, we get

$$\tau^*_{rr} = \frac{k}{c}(2 - c)\left[1 - \left(\frac{A}{a}\right)^c\right]\left(\frac{a}{A}\right)^c, \quad (a \leqslant r \leqslant A) \qquad \ldots(3.10)$$

and

$$\tau^*_{rr} = \frac{k}{c}(2 - c)\left[1 - \left(\frac{b}{A}\right)^{c/(1-c)}\right]\left(\frac{A}{b}\right)^{c/(1-c)}, \quad (A \leqslant r \leqslant b). \qquad \ldots(3.11)$$

Since τ_{rr} must be continuous across $r = A$, the radius of the neutral surface, by equating (3.10) and (3.11) it is found that

$$A = a^{(1-c)/(2-c)} \cdot b^{1/(2-c)}. \qquad \ldots(3.12)$$

We note that radius of neutral surface varies with c where $0 \leqslant c \leqslant 1$.

Using eqns. (3.8), (3.10) and (3.11) in eqn. (2.5), we get $\tau_{\theta\theta}$ for the region of tension as

$$\tau_{\theta\theta} = (1 - c)\left(\frac{A}{r}\right)^c \tau^*_{rr} + \frac{k}{c}(2 - c)\left[1 - (1 - c)\left(\frac{A}{r}\right)^c\right]. \quad ...(3.13)$$

Similarly, we have for the region of compression

$$\tau_{\theta\theta} = \frac{1}{1 - c}\left(\frac{r}{A}\right)^{c/(1-c)} \tau^*_{rr} + \frac{k}{c}(2 - c)\left[1 - \frac{1}{1 - c}\left(\frac{r}{A}\right)^{c/(1-c)}\right] \quad ...(3.14)$$

where

$$\tau^*_{rr} = \frac{k}{c}(2 - c)\left[\left(\frac{a}{b}\right)^{c/(2-c)} - 1\right] \quad ...(3.15)$$

The elastic-plastic transitional stresses are given by eqns. (3.8), (3.13) and (3.15) for the region of tension ($a \leqslant r \leqslant A$) and eqns. (3.9), (3.14) and (3.15) represents the stresses for the region of compression ($A \leqslant r \leqslant b$).

For fully plastic state, letting $\sigma \to \frac{1}{2}$ i.e., $c \to 0$ (Seth 1963b), in eqn. (3.12), the neutral surface becomes

$$A = \sqrt{ab} \quad ...(3.16)$$

Condition (3.16) is generally found in the classical theory by putting Tresca's yield condition in equilibrium eqn. (2.5).

From eqns. (3.8) and (3.13) we have the plastic stresses in the region of tension

$$\tau_{rr} = 2k \log \frac{r}{a} \quad ...(3.17)$$

$$\tau_{\theta\theta} = 2k + \tau_{rr} \equiv 2k\left(1 + \log \frac{r}{a}\right) \quad ...(3.18)$$

and from eqns. (3.9) and (3.14) for the region of compression

$$\tau_{rr} = 2k \log \frac{b}{r} \quad ...(3.19)$$

$$\tau_{\theta\theta} = \tau_{rr} - 2k \equiv -2k\left(1 - \log \frac{b}{r}\right) \quad ...(3.20)$$

From eqns. (3.18) and (3.20), we note that Tresca's yield condition comes out from the analysis itself.

Also from eqn. (2.4), we have

$$\tau_{zz} = \frac{1}{2}(\tau_{rr} + \tau_{\theta\theta}) + K_1 \quad ...(3.21)$$

where

$$K_1 = 3ke_{zz} \equiv \tfrac{3}{2}\, k\,[1 - (1 - \alpha)^2] \equiv 3kK.$$

Applying the boundary condition (2.8) in eqn. (3.21) we found that $K_1 = 0$, which implies that $e_{zz} = 0$. It leads to the result that α vanishes. This means that the fibres parallel to z'-axis suffer no change (plane strain condition). It can be readily seen that the boundary condition (2.8) is also satisfied.

The bending moment M is given by

$$M = -\int_a^b r\tau_{\theta\theta}\, dr = \frac{k}{2}\,(b - a)^2. \qquad \text{...(3.22)}$$

The results (3.16) to (3.20) and (3.22) for fully plastic state are the same as obtained by Shaffer and House (1955) by assuming $\tau_{\theta\theta} > \tau_{rr}$ in the region of tension, $\tau_{rr} > \tau_{\theta\theta}$ in the region of compression and Tresca's yield condition.

4. Discussion

When a material passes from elastic state to the plastic state, transition takes place. Since this transition is nonlinear and difficult to investigate, workers have assumed certain adhoc assumptions which may or may not exist. It has been shown in this paper that by using Seth's transition theory, there is no need to assume different conditions as given in section 1, but comes out from the analysis themselves, which can be seen from eqns. (2.7), (3.16), (3.18) and (3.20). To understand the inter-relationship between some of the parameters, numerical calculations are carried out and the results obtained are shown graphically in Figs. 1 and 2. It can be seen that the radius of the neutral surface oscillates in between the transitional period of the elastic-plastic state.

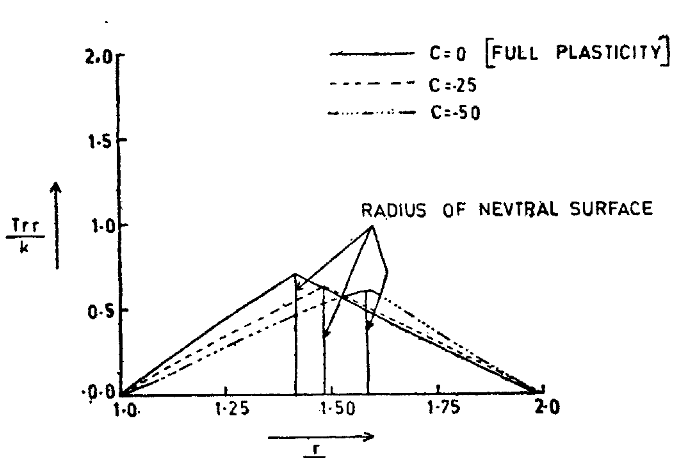

FIG. 1. Radial stress distribution for $b/a = 2.0$.

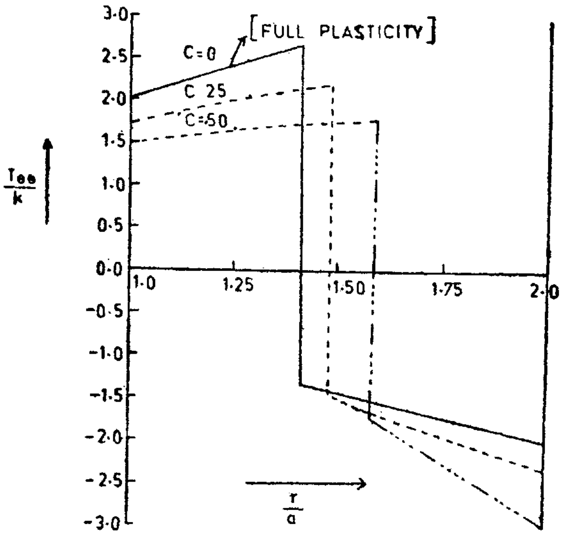

FIG. 2. Circumferential stress distribution for $b/a = 2.0$.

In Fig. 1 the radial stress reaches its maximum value at the neutral surface for completely plastic state, the magnitude of which is obtained by combining eqns. (3.16) and (3.17) equal to $\tau_{rr} = k \log (b/a)$.

The circumferential stress distribution is shown in Fig. 2. The fully plastic solution is in agreement with the previous works of Shaffer and House (1955), Lubahn and Sach (1950) and Hill (1950).

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