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DYNAMIC EXPANSION OF A SPHERICAL CAVITY IN A NON-HOMOGENIOUS MEDIUM

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In the present paper the dynamic expansion of a spherical cavity in a non-homogeneous elastic solid, due to the action of suddenly applied pressure which is supposed to be high is discussed. The resulting velocity is large but the displacement for small values of time is small. Under these assumptions the equation of motion is integrated to obtain the stresses. The problem is reduced to an integral equation which is solved to obtain the velocity of the inner surface of the cavity as a function of the inner radius.

INTRODUCTION

The problem of dynamic expansion of a spherical cavity in an elastic material has been treated by Hopkins and Cox (1957) and Hunter (1958). In the present paper we discuss the dynamic expansion of a spherical cavity in a non-homogeneous elastic solid. The problem has been reduced to the solution of an integral equation.

FORMULATION OF THE PROBLEM AND THE EQUATION OF MOTION

The motion being spherically symmetric the only non-zero displacement component is radial and the stresses, displacements and velocities are functions of r and t only. As the radius a of the cavity at time t is supposed to increase with time, the motion may be studied using a as parameter.

The equation of conservation of mass is

$$\frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v) + \frac{\partial \rho}{\partial t} = 0 \qquad \ldots(1)$$

where ρ is the density and let us take $\rho = \rho_0\, r^n/a_0^n$, and v is the particle velocity at any time t.

Now from eqn. (1) we get

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho_0 \frac{r^{n+2}}{a_0^n} v\right) = 0$$

$$\therefore \qquad \frac{\partial}{\partial r}(r^{n+2}v) = 0.$$

Integrating with respect to r we get

$$r^{n+2}v = \text{const.} = \text{function of time only}$$

$$= a^{n+2}\dot{a}.$$

$$\therefore \qquad v = \frac{a^{n+2}}{r^{n+2}}\dot{a}$$

$$\frac{\partial u}{\partial t} = \frac{a^{n+2}}{r^{n+2}}\frac{da}{dt}.$$

Integrating we get

$$u = \frac{a^{n+3} - a_0^{n+3}}{(n+3)\,r^{n+2}} \qquad \text{...(2)}$$

where u is the particle displacement and a_0 is the initial cavity radius.

The equation of motion is

$$\frac{\partial \sigma_r}{\partial r} + \frac{2}{r}(\sigma_r - \sigma_\theta) = \rho\frac{d^2u}{dt^2} \qquad \text{...(3)}$$

where ρ is the density, $\frac{d^2u}{dt^2}$ is the accelaration, σ_r, σ_θ are the normal stresses in the r and θ directions.

The stress-strain relation is

$$\left.\begin{aligned} e_{rr} &= \frac{1}{E}[\sigma_r - \nu(\sigma_\theta + \sigma_\varphi)] \\ e_{\theta\theta} &= \frac{1}{E}[\sigma_\theta - \nu(\sigma_\varphi + \sigma_r)]. \end{aligned}\right\} \qquad \text{...(4)}$$

For spherical symmetry $\sigma_\theta = \sigma_\varphi$ and for incompressibility, Poisson's ratio $\nu = \frac{1}{2}$, therefore eqns. (4) become

$$e_{rr} = \frac{\partial u}{\partial r} = \frac{1}{E}(\sigma_r - \sigma_\theta)$$

$$E\frac{\partial u}{\partial r} = (\sigma_r - \sigma_\theta) \qquad \text{...(5)}$$

$$e_{\theta\theta} = \frac{u}{r} = \frac{1}{E}(\sigma_\theta - \sigma_r)$$

$$\therefore \quad E\frac{u}{r} = (\sigma_\theta - \sigma_r) \qquad \ldots(6)$$

where E is the Young's modulus.

For non-homogeneity

$$E = E_0 \frac{r^n}{a_0^n}, \quad \rho = \rho_0 \frac{r^n}{a_0^n}.$$

Therefore from (6) and (2) we get

$$\sigma_\theta - \sigma_r = 2E\frac{u}{r}$$

$$= 2E_0 \frac{r^n}{a_0^n} \frac{a^{n+3} - a_0^{n+3}}{(n+3)\, r^{n+3}}, \quad (n \neq -3)$$

Substituting the above in eqn. (3) we get

$$\frac{\partial\sigma_r}{\partial r} - \frac{4E_0}{(n+3)\, a_0^n} \frac{a^{n+3} - a_0^{n+3}}{r^4} = \frac{\rho_0}{a_0^n}\Big[\{a^{n+2}\ddot{a} + (n+2)\, a^{n+1}\dot{a}^2\}/r^2 - (n+2)\frac{a^{2(n+2)}}{r^{n+5}}\dot{a}^2\Big].$$

Integrating w.r.t. r we get

$$\sigma_r + 4E_0 \frac{(a^{n+3} - a_0^{n+3})\, r^{-3}}{(n+3)\, a_0^n \,.\, 3} = \frac{\rho_0}{a_0^n}\Big[\{a^{n+2}\ddot{a} + (n+2)\, a^{n+1}\dot{a}^2\}\frac{r^{-1}}{-1} + \frac{(n+2)\, a^{2(n+2)}}{(n+4)\, r^{n+4}} + f(t)$$

Now $f(t) = 0$ since radial stress tends to zero as $r \to \infty$ supposing $n > -4$.

$$\sigma_r = -\frac{4E_0(a^{n+3} - a_0^{n+3})}{3(n+3)\, a_0^n\, r^3} + \frac{\rho_0}{a_0^n}\Big[\frac{(n+2)\, a^{2(n+2)}}{(n+4)\, r^{n+4}}\dot{a}^2 - \{a^{n+2}\ddot{a} + (n+2)\, a^{n+1}\dot{a}^2\}/r\Big]. \qquad \ldots(7)$$

Boundary condition is

$$(\sigma_r)_{r=a} = -p$$

where p is the cavity pressure. From (7) and boundary condition we get

$$-p = -\frac{4E_0}{3(n+3)\,a_0^n}\,\frac{a^{n+3}-a_0^{n+3}}{a^3} + \frac{\rho_0}{a_0^n}\left[\frac{(n+2)}{(n+4)}\,\frac{a^{2(n+2)}}{a^{n+4}}\,\dot{a}^2 - \{a^{n+2}\ddot{a}\right.$$

$$\left. + (n+2)\,a^{n+1}\dot{a}^2\}/a\right]$$

$$\therefore \quad p = \frac{4E_0}{3(n+3)\,a_0^n}\,\frac{a^{n+3}-a_0^{n+3}}{a^3} + \frac{\rho_0}{a_0^n}\left[\frac{(n+3)\,(n+2)}{(n+4)}\,a^n\dot{a}^2 + a^{n+1}\ddot{a}\right].$$

Multiplying both sides by a^2da and integrating from a_0 to a,

$$\int_{a_0}^{a} pa^2da = \frac{4E_0}{3(n+3)\,a_0^n}\int_{a_0}^{a}\frac{a^{n+3}-a_0^{n+3}}{a}\,da + \frac{\rho_0}{a_0^n}\left[\frac{(n+3)\,(n+2)}{n+4}\right.$$

$$\left.\times \int_{a_0}^{a} a^{n+2}\dot{a}^2da + \int_{a_0}^{a} a^{n+3}\ddot{a}\;da\right].$$

Supposing p to be practically constant, say p_0, we get approximately

$$p_0\,\frac{(a^3-a_0^3)}{3} = \frac{4E_0}{3(n+3)\,a_0^n}\left[\frac{a^{n+3}-a_0^{n+3}}{n+3} - a_0^{n+3}\log\frac{a}{a_0}\right] + \frac{\rho_0}{a_0^n}$$

$$\times\left[\frac{(n+3)\,(n+2)}{n+4}\int_{a_0}^{a} a^{n+2}\dot{a}^2\;da + a^{n+3}\,\frac{\dot{a}^2}{2}\right.$$

$$\left. -(n+3)\int_{a_0}^{a} a^{n+2}\,\frac{\dot{a}^2}{2}\;da\right]$$

or retaining second order terms in expansion of log (a/a_0)

$$p_0\,\frac{(a^3-a_0^3)}{3} = \frac{4E_0}{3(n+3)\,a_0^n}\left[\frac{a^{n+3}-a_0^{n+3}}{n+3} - \frac{a_0^{n+3}}{n+3}\left\{\frac{a^{n+3}-a_0^{n+3}}{a_0^{n+3}} - \right.\right.$$

(*equation continued on p.* 1246)

$$- \tfrac{1}{2}\left(\frac{a^{n+3}-a_0^{n+3}}{a_0^{n+3}}\right)^2 \Bigg\}\Bigg] + \frac{\rho_0}{a_0^n}\left[a^{n+3}\frac{\dot a^2}{2} + \frac{n(n+3)}{2(n+4)} \times \int_{a_0}^{a} a^{n+2}\dot a^2\, da\right]$$

or

$$p_0\frac{(a^3-a_0^3)}{3} = \frac{2E_0 a_0^{n+3}}{3(n+3)^2 a_0^n}\left(\frac{a^{n+3}-a_0^{n+3}}{a_0^{n+3}}\right)^2 + \frac{\rho_0}{a_0^n} a^{n+3}\frac{\dot a^2}{2} + \frac{n(n+3)\rho_0}{2(n+4)a_0^n}\int_{a_0}^{a} a^{n+2}\dot a^2 da,$$

or

$$\frac{\rho_0}{2a_0^n} a^{n+3}\dot a^2 = \frac{p_0(a^3-a_0^3)}{3} - \frac{2E_0}{3(n+3)^2 a_0^{2n+3}}(a^{n+3}-a_0^{n+3})^2 - \frac{\rho_0 n(n+3)}{2a_0^n(n+4)}\int_{a_0}^{a} a^{n+2}\dot a^2 da. \qquad \text{...(8)}$$

Let us write eqn. (8) in the form

$$c_1 a^{n+3} f(a) = d_1(a^3-a_0^3) + d_2(a^{n+3}-a_0^{n+3})^2 + c_2\int_{a_0}^{a} t^{n+2} f(t)\, dt$$

where

$$f(a) = \dot a^2,\ c_1 = \frac{\rho_0}{2a_0^n},\ \ d_1 = \frac{p_0}{3},\ c_2 = -\frac{\rho_0 n(n+3)}{2a_0^n(n+4)},\ \ d_2 = \frac{-2E_0}{3(n+3)^2 a_0^{2n+3}}.$$

For the solution of (8) let us suppose $\dot a^2$ to be the following function $f(a)$ where

$$f(a) = \frac{\alpha_1}{a} + \alpha_2 a^{n+3} + \alpha_3 + \frac{\alpha_4}{a^n} \qquad \text{...(9)}$$

and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are constants to be determined from eqn. (8).

$$\therefore \quad c_1 a^{n+3}\left[\frac{\alpha_1}{a} + \alpha_2 a^{n+3} + \alpha_3 + \frac{\alpha_4}{a^n}\right] = d_1(a^3-a_0^3) + d_2(a^{n+3}-a_0^{n+3})^2 +$$

(*equation continued on p.* 1247)

$$+ c_2 \int_{a_0}^{a} t^{n+2}\left(\frac{\alpha_1}{t} + \alpha_2 t^{n+3} + \alpha_3 + \frac{\alpha_4}{t^n}\right) dt$$

The above equation reduces to

$$c_1 a^{n+3}\left[\frac{\alpha_1}{a} + \alpha_2 a^{n+3} + \alpha_3 + \frac{\alpha_4}{a^n}\right]$$

$$= d_1(a^3 - a_0^3) + d_2(a^{n+3} - a_0^{n+3})^2 + c_2\left[\alpha_1\left(\frac{a^{n+2} - a_0^{n+2}}{n+2}\right)\right.$$

$$\left. + \alpha_2\left(\frac{a^{2n+6} - a_0^{2n+6}}{2n+6}\right) + \alpha_3\left(\frac{a^{n+3} - a_0^{n+3}}{n+3}\right) + \alpha_4\left(\frac{a^3 - a_0^3}{3}\right)\right]$$

on the assumption that $3, n+3, n+2$ and $2n+6$ are all different i.e. for values of n other than $0, 1, -4, -2, -3/2$.

Equating coefficients of a^3, a^{n+2}, a^{n+3}, a^{2n+6} we get

$$\left.\begin{aligned}
c_1\alpha_4 &= d_1 + \frac{c_2\alpha_4}{3} \rightarrow \alpha_4 = \frac{2p_0 a_0^n (n+4)}{\rho_0(n^2+6n+12)} \\
c_1\alpha_1 &= \frac{c_2\alpha_1}{n+2} \rightarrow \alpha_1 = 0 \\
\alpha_3 &= \frac{4E_0(n+4)}{3(n+3)^2(n+2)\rho_0} \\
c_1\alpha_2 &= d_2 + \frac{c_2\alpha_2}{2n+6} \rightarrow \alpha_2 = -\frac{4E_0(n+4)(2n+6)}{3\rho_0(n+3)^3 a_0^{n+3}(3n+8)}
\end{aligned}\right\} \quad \ldots(10)$$

Therefore, from (9) and (10) we get

$$f(a) = \dot{a}^2 = \frac{-4E_0(n+4)(2n+6)}{3\rho_0(n+3)^3 a_0^{n+3}(3n+8)} \frac{a^{n+3}}{a_0^{n+3}} + \frac{4E_0(n+4)}{3\rho_0(n+3)^2(n+2)}$$

$$+ \frac{2p_0(n+4)}{\rho_0(n^2+6n+12)} \frac{a_0^n}{a^n}$$

$$\therefore \quad \frac{\rho_0\dot{a}^2}{E_0} = \frac{4(n+4)}{3(n+3)^2(n+2)} + \frac{2p_0(n+4)}{E_0(n^2+6n+12)} \frac{a_0^n}{a^n}$$

$$- \frac{4(n+4)(2n+6)}{3(n+3)^3(3n+8)} \frac{a^{n+3}}{a_0^{n+3}}.$$

The values of $\dot{a}^2$ for different n have been evaluated and shown as follows:

For $n = 0.5, \frac{p_0}{E_0} = 0.5$

$\frac{a}{a_0}$	1.1	1.3	1.5
$\rho_0 \frac{\dot{a}^2}{E_0}$	0.333323	0.196423	0.010625

For $n = 0.75, \frac{p_0}{E_0} = 1$

$\frac{a}{a_0}$	1.1	1.3	1.5	1.6
$\rho \frac{\dot{a}^2}{E_0}$	0.556505	0.386043	0.172561	0.043078

For $n = 1.5, \frac{p_0}{E_0} = 2$

$\frac{a}{a_0}$	1.1	1.3	1.5	1.6
$\rho \frac{\dot{a}^2}{E_0}$	0.834678	0.553169	0.259276	0.090686

References

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