Indian J. pure appl. Math., 11(9) : 1249-1257, September 1980

PROPAGATION, REFLECTION AND TRANSMISSION OF LONGITUDINAL WAVES IN NON-HOMOGENEOUS FIVE-PARAMETER VISCOELASTIC RODS

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(*Received* 20 *July* 1979)

In this paper the asymptotic methods have been employed to study the propagation of longitudinal waves in linear non-homogeneous thin five-parameter viscoelastic rods. Methods for treating reflection at the free end of the finite rod and reflection and transmission at an interface between two media in the semi-infinite bi-viscoelastic rod, are also presented.

1. INTRODUCTION

Many of the analytical solutions for one-dimensional viscoelastic waves in homogeneous and isotropic media can be found in the treatises by Christensen (1971), Bland (1960) and Kolsky (1963) and the references contained therein. To overcome the transform inversion difficulties in the homogeneous viscoelastic wave problems as far as the Laplace transform treatment concerns, the asymptotic and approximate solutions are also reported in the same monographs. In the present paper, the asymptotic methods have been employed to study the propagation of longitudinal waves in linear non-homogeneous thin five-parameter viscoelastic rods, due to suddenly applied longitudinal traction on the end and thereafter steadily maintained. It has been found that a reflected wave is produced at the free end of the finite rod while reflected and transmitted waves are produced at an interface between two media in the semi-infinite bi-viscoelastic rod. The solutions are obtained in the form of asymptotic expansions about the time of arrival of the wave-front. The rods are supposed to be initially unstressed and at rest.

The technique used in this paper was formulated by Friedlander (1947) for general progressing waves in the homogeneous elastic media. Achenbach and Reddy (1967) have used this technique to study the propagation of a stress discontinuity in a linear homogeneous thin integral type viscoelastic semi-infinite rod. Park and Reiss (1970) then applied this technique to study oscillatory impact of a standard non-homogeneous viscoelastic rod. Moodie (1973) has obtained asymptotic solutions for radially symmetric transient stress waves in non-homogeneous four-parameter viscoelastic media.

2. Formulation

We formulate the one-dimensional wave problem by taking the end of the semi-infinite rod under consideration as $x = 0$. The coordinate x is measured positive in the direction of the axis of the rod, and the time is denoted by t . σ, e and u respectively denote the only non-zero components of stress, strain and displacement.

We consider the five-parameter model which consists of a two parameter Maxwell element in series with a three parameter model constructed from a parallel combination of a viscous and a two parameter Maxwell element. This model exhibits an elastic, a viscous, and a retarded elastic response to normal stress. The behaviour of the viscoelastic material conforming to this five-parameter model, is represented by the following stress-strain law :

$$\left.\begin{aligned} e &= e_1 + e_2 + e_3 \\ \sigma &= G_1 \cdot e_1 \\ \frac{\partial \sigma}{\partial t} + \frac{G_2}{\eta_2}\,\sigma &= \eta_2' \frac{\partial^2 e_2}{\partial t^2} + G_2\left(1 + \frac{\eta_2'}{\eta_2}\right)\cdot\frac{\partial e_2}{\partial t} \\ \sigma &= \eta_3 \cdot \frac{\partial e_3}{\partial t} \end{aligned}\right\} \qquad \ldots(2.1)$$

where σ is the normal stress and e is the corresponding overall normal strain of the five-parameter model; e_1, e_2 and e_3 are the normal strains associated with the Maxwell spring, the three parameter model and the Maxwell dashpot, respectively. Also $G_1 = \lambda_1 + 2\mu_1$ and $G_2 = \lambda_2 + 2\mu_2$ are the moduli of elasticity associated with the Maxwell and the three parameter element respectively; and η_2, η_2' and η_3 are Newtonian viscosity coefficients associated with these elements. The moduli of elasticity and Newtonian viscosity coefficients are all taken as functions of x in the non-homogeneous case considered here.

Eliminating e_1, e_2 and e_3 from eqns. (2.1), we obtain the stress-strain relation for our viscoelastic body in the form

$$\begin{aligned} \frac{\partial^2 \sigma}{\partial t^2} &+ G_1\left\{\frac{1}{\eta_2'} + \frac{1}{\eta_3} + \frac{G_2}{G_1 \eta_2'}\left(1 + \frac{\eta_2'}{\eta_2}\right)\right\}\frac{\partial \sigma}{\partial t} \\ &+ G_1\left\{\frac{G_2}{\eta_2'\eta_2} + \frac{G_2}{\eta_2'\eta_3}\left(1 + \frac{\eta_2'}{\eta_2}\right)\right\}\sigma \\ &= G_1 \frac{\partial^2 e}{\partial t^2} + \frac{G_1 G_2}{\eta_2'}\left(1 + \frac{\eta_2'}{\eta_2}\right)\frac{\partial e}{\partial t}. \end{aligned} \qquad \ldots(2.2)$$

The equation of motion and the strain-displacement relation are

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \qquad \ldots(2.3)$$

$$e = \partial u/\partial x \qquad \text{...(2.4)}$$

where $\rho = \rho(x)$ is the variable density of the material.

Equations (2.2), (2.3) and (2.4) lead to

$$\frac{\partial^3\sigma}{\partial t^3} + \left\{\frac{G_1}{\eta_2'} + \frac{G_1}{\eta_3} + \frac{G_2}{\eta_2'}\left(1 + \frac{\eta_2'}{\eta_2}\right)\right\}\frac{\partial^2\sigma}{\partial t^2}$$

$$+ \frac{G_1 G_2}{\eta_2'}\left\{\frac{1}{\eta_2} + \frac{1}{\eta_3}\left(1 + \frac{\eta_2'}{\eta_2}\right)\right\}\frac{\partial\sigma}{\partial t}$$

$$= \frac{G_1}{\rho}\left\{\frac{\partial}{\partial t} + \frac{G_2}{\eta_2'}\left(1 + \frac{\eta_2'}{\eta_2}\right)\right\}\left\{\frac{\partial}{\partial x} - \frac{\rho'}{\rho}\right\}\frac{\partial\sigma}{\partial x}. \qquad \text{...(2.5)}$$

Now the stress field $\sigma(x, t)$ has to be determined from eqn. (2.5) under the prescribed boundary conditions.

3. Method of Solution

We assume that the solution $\sigma(x, t)$ of eqn. (2.5) may be represented by the series (Friedlander 1947)

$$\sigma(x, t) \simeq \sum_{n=0}^{\infty} A_n(x)\, F_n\,[t - h(x)],\ A_0 \neq 0 \qquad \text{...(3.1)}$$

where the F_n's are related by

$$F_n' = F_{n-1},\quad n = 1, 2, \ldots \qquad \text{...(3.2)}$$

and it is assumed that $A_n \equiv 0$ for $n < 0$. We further assume that the derivatives of σ may be obtained by term-wise differentiation of (3.1). The prime in (3.2) denotes differentiation with respect to the argument of the function concerned, and (3.2) enables us to relate all of the F_n's to F_0 by successive integrations. To obtain the solution of eqn. (2.5) in the form (3.1), we find (Karal and Keller 1959) that the phase function $h(x)$ satisfies the eikonal equation of geometrical optics

$$\left(\frac{dh(x)}{dx}\right)^2 = \frac{\rho}{G_1} = \frac{1}{c^2} \qquad \text{...(3.3)}$$

where $c = c(x)$ is the variable wave speed for elastic longitudinal waves in a medium whose modulus is G_1.

From eqns. (2.5) and (3.1), the amplitude functions $A_n(x)$ satisfy the equations

$$2h'(x)\, A_n'(x) + \rho\left(\frac{1}{\eta_2'} + \frac{1}{\eta_3}\right)A_n(x) - \frac{\rho'}{\rho}\,h(x)\,A_n(x) + h''(x)\,A_n(x) = P_n,$$

$$n = 0, 1, 2, \ldots \qquad \text{...(3.4)}$$

where

$$\left.\begin{aligned} P_n &\equiv A''_{n-1} - \left(\frac{\rho'}{\rho} + 2kh'\right) A'_{n-1} - \left\{\rho\left(\frac{G_2}{\eta_2\eta_2'} + \frac{k}{\eta_3}\right) + kh'' \right. \\ &\quad \left. - k\frac{\rho'}{\rho} h'\right\} A_{n-1} + kA''_{n-2} - k\frac{\rho'}{\rho} A'_{n-2} \\ k(x) &= \frac{G_2}{\eta_2'}\left(1 + \frac{\eta_2'}{\eta_2}\right). \end{aligned}\right\} \quad \ldots(3.5)$$

Integrating eqn. (3.3) along the x-axis, we obtain

$$h(x) = h(0) \pm \int_0^x \frac{ds}{c(s)} \quad \ldots(3.6)$$

where the plus sign is associated with waves travelling in the positive direction of x and the minus sign is associated with waves travelling in the negative direction of x.

The general solution of eqn. (3.4) is obtained as

$$A_n(x) = A_n(0)\left\{\frac{l(x)}{l(0)}\right\}^{1/2} \exp\left\{\mp \int_0^x m(s)\, ds\right\}$$

$$\pm \tfrac{1}{2} \int_0^x c(s)\left\{\frac{l(x)}{l(s)}\right\}^{1/2} \exp\left\{\pm \int_x^s m(z)\, dz\right\} P_n^{\pm}(s)\, ds,$$

$$n = 0, 1, 2, \ldots \quad \ldots(3.7)$$

where

$$l(x) = \rho c \quad \text{and} \quad m(x) = \frac{\rho c}{2}\left\{\frac{1}{\eta_2'} + \frac{1}{\eta_3}\right\}.$$

In eqn. (3.7) the upper signs are associated with waves travelling in the positive direction of x and the lower signs are associated with waves travelling in the negative direction of x. The expressions $P_n^{\pm}$ are obtained from eqns. (3.5) and (3.6), that is, when the plus sign is used with $h(x)$ in the expression P_n we call it P_n^{+}, and when the minus sign is used, P_n^{-}. Thus P_n^{+} is associated with waves travelling in the positive direction of x and P_n^{-} with waves travelling in the negative direction of x.

Consider now an impulse of magnitude σ_0 suddenly applied at the end $x = 0$ of the rod and thereafter steadily maintained, that is,

$$\sigma(0, t) = \sigma_0 H(t). \qquad ...(3.8)$$

From eqns. (3.1) and (3.8) we have

$$\sum_{n=0}^{\infty} A_n(0)\, F_n\, [t - h(0)] = \sigma_0 H(t). \qquad ...(3.9)$$

Thus we choose (Moodie 1973)

$$A_n(0) = \begin{cases} \sigma_0 & \text{if } n = 0, \\ 0 & \text{if } n < 0 \text{ or } n > 0, \end{cases} \qquad ...(3.10)$$

$$h(0) = 0 \text{ and } F_0 = H(t). \qquad ...(3.11)$$

The solution of eqn. (2.5), for waves travelling in the positive direction of x generated by the boundary stress (3.8), is given by

$$\sigma(x, t) \simeq \sum_{n=0}^{\infty} A_n(x) \frac{(t - h(x))^n}{n!} H(t - h(x)) \qquad ...(3.12)$$

$$h(x) = \int_0^x \frac{ds}{c(s)} \qquad ...(3.13)$$

where the $A_n(x)$ are given recursively by (3.7) (with the upper signs) in conjunction with (3.10). The first-term approximation to (3.12) is

$$\sigma(x, t) \simeq \sigma_0 \left\{\frac{l(x)}{l(0)}\right\}^{1/2} \exp\left\{-\int_0^x m(s)\, ds\right\} H(t - h(x)) \qquad ...(3.14)$$

where $h(x)$ is given by (3.13).

The expression (3.14) represents a transient stress wave which starts from the end $x = 0$ with amplitude σ_0 and moves in the positive direction of x with velocity $c(x)$. It is modulated by the factor

$$\left\{\frac{l(x)}{l(0)}\right\}^{1/2} \exp\left\{-\int_0^x m(s)\, ds\right\}.$$

Further terms in the approximate solution may be obtained recursively from (3.7).

The solution (3.12) applies until the wave moving in the positive direction of x strikes either an interface (in the case of a composite rod) or an end (in the case of a finite rod). We will show that reflected waves are produced at the other end of the finite rod while both reflected and transmitted waves are produced at an interface between two dissimilar media.

4. The Finite Rod

We consider a thin non-homogeneous finite viscoelastic rod $0 \leqslant x \leqslant x_1$, with the stress-free end $x = x_1$. When the end $x = 0$ of the rod is subjected to a suddenly rising traction of magnitude σ_0 and thereafter steadily maintained as described in the preceding section by eqn. (3.8), a transient stress wave is produced which leaves the end $x = 0$ at time $t = 0$ and travels with the speed $c(x)$ in the positive direction of x. The asymptotic representation of this wave is given by (3.12) with the amplitude functions $A_n(x)$ being given by (3.7) and (3.10), where the upper signs in (3.7) are chosen to correspond to a wave moving in the positive direction of x. From now onwards we will denote this wave by σ^i and its amplitude and phase functions by $A_n^i(x)$ and $h^i(x)$, respectively, implying that it is the wave which is incident at the stress-free end $x = x_1$.

We now assume that at the stress-free end there is produced a reflected wave σ^r which can be represented by

$$\sigma^r \simeq \sum_{n=0}^{\infty} A_n^r(x)\, F_n\,[t - h^r(x)],\; A_n^r(x) \equiv 0 \text{ for } n < 0. \qquad \ldots(4.1)$$

The amplitude functions $A_n^r(x)$ in (4.1) satisfy the transport equation (3.4). Its solution for a reflected wave which leaves $x = x_1$ and proceeds in the negative direction of x is given by (3.7) with the lower signs, and x_1 replacing 0. The phase function $h^r(x)$ is given by eqn. (3.6) with the minus sign, and x_1 replacing 0. We justify our assumption that a reflected wave is produced at $x = x_1$ by demonstrating that the boundary condition on x_1 can be satisfied by $\sigma = \sigma^i + \sigma^r$. The boundary condition will also give the necessary initial conditions for the eikonal and transport equations thereby enabling us to formally determine the reflected wave.

At $t = h^i(x_1)$ the incident wave has arrived at the end $x = x_1$, and applying the condition of the stress-free end $x = x_1$ we obtain

$$A_n^r(x_1) = -A_n^i(x_1),\; n = 0, 1, 2, \ldots \qquad \ldots(4.2)$$

$$h^r(x_1) = h^i(x_1). \qquad \ldots(4.3)$$

Then, because $A_n^i(x_1)$ and $h^i(x_1)$ are known from the work of the preceding section, the reflected wave may be completely determined. It is given by (4.1) with A_n^r and h^r being given by

$$A_n^r(x) = -A_n^i(x_1)\left\{\frac{l(x)}{l(x_1)}\right\}^{1/2} \exp\left\{\int_{x_1}^{x} m(s)\,ds\right\}$$

$$-\tfrac{1}{2}\int_{x_1}^{x} c(s)\left\{\frac{l(x)}{l(s)}\right\}^{1/2} \exp\left\{-\int_{x}^{s} m(z)\,dz\right\} P_n^-(s)\,ds,\ n = 0, 1, \ldots \quad \ldots(4.4)$$

and

$$h^r(x) = \left\{\int_0^{x_1} - \int_{x_1}^{x}\right\}\left\{\frac{ds}{c(s)}\right\}. \quad \ldots(4.5)$$

The solution (4.1) for σ^r will then apply for $h^i(x_1) = h^r(x_1) < t < h^r(0)$. At time $t = h^r(0)$ the reflected wave σ^r arrives at $x = 0$ and the boundary conditions there must be satisfied afresh. The process is then repeated and in this way any number of reflections can be treated in a straightforward manner.

The first-term approximation to the reflected wave is

$$\sigma^r(x, t) \simeq -\sigma_0\left\{\frac{l(x)}{l(0)}\right\}^{1/2} \exp\left\{-\left(\int_0^{x_1} + \int_x^{x_1}\right) m(s)\,ds\right\} H(t - h^r(x))$$

where $h^r(x)$ is given by eqn. (4.5). Further terms in this approximate solution may be obtained recursively from (4.4).

5. The Bi-viscoelastic Semi-infinite Rod

We consider here the semi-infinite non-homogeneous viscoelastic rod $0 \leqslant x < \infty$, composed of two dissimilar parts which we call medium 'a' and medium 'b'. The interface between these two media is at $x = \bar{x}$. We denote the wave speed and density of the two media by c_a, c_b, and ρ_a, ρ_b respectively. The functions evaluated at the interface will be denoted by a bar, that is, $h^i(\bar{x}) = \bar{h}^i$. The condition to be satisfied at the interface requires the continuity of stress and velocity.

At the end $x = 0$ of the rod we apply the suddenly rising longitudinal traction and thereafter steadily as maintained discussed in the preceding section. A transient stress wave is produced which leaves the end $x = 0$ at time $t = 0$ and travels with the speed $c_a(x)$ in the positive direction of x through medium a. This wave has been discussed in section 3 and the solution given there will be valid until such time as the wave moving in the positive direction of x arrives at the interface $x = \bar{x}$ between the media a and b. We assume that when the incident wave strikes the interface $x = \bar{x}$, there is produced both a reflected wave which travels in the negative direction of x into medium a, and a transmitted wave which proceeds from the interface into medium b. Due to linearity of the equations, the stress field can be obtained by superposition. We assume that

$$\sigma_a = \sum_{n=0}^{\infty} A_n^i F_n [t - h^i] + \sum_{n=0}^{\infty} A_n^r F_n [t - h^r] \qquad ...(5.1)$$

$$\sigma_b = \sum_{n=0}^{\infty} A_n^t F_n [t - h^t] \qquad ...(5.2)$$

$$v_a = \sum_{n=0}^{\infty} B_n^i F_n [t - h^i] + \sum_{n=0}^{\infty} B_n^r F_n [t - h^r] \qquad ...(5.3)$$

$$v_b = \sum_{n=0}^{\infty} B_n^t F_n [t - h^t] \qquad ...(5.4)$$

where $v = \frac{\partial u}{\partial t}$ is the particle velocity; and the superscripts i, r and t refer to incident, reflected, and transmitted waves, respectively. The form for the velocity field in (5.3) implies that the A_n^j and B_n^j are related by

$$B_n^j = -\frac{1}{\rho} (h^{j'} A_n^j - A_{n-1}^{j'}),\ n = 0, 1, 2, ...,\ j = i, r, t \qquad ...(5.5)$$

where $h^{j'} = c^{-1}$ for the waves moving in the positive direction of x and $h^{j'} = -c^{-1}$ for the waves moving in the negative direction of x. Equation (5.5) yields the well-known result (Moodie 1973) from the conservation of momentum across a steadily moving discontinuity, that is, $A_0^j = \rho c B_0^j$.

Inserting (5.1), (5.3) and (5.5) in the boundary conditions $\bar{\sigma}_a = \bar{\sigma}_b$ and $\bar{v}_a = \bar{v}_b$, we find that

$$\bar{h}^i = \bar{h}^r = \bar{h}^t \qquad ...(5.6)$$

$$\bar{A}_n^r - \bar{A}_n^t = -\bar{A}_n^i \qquad ...(5.7)$$

and

$$\bar{A}_n^r + \bar{I}\bar{A}_n^t = \bar{A}_n^i - \bar{f}_{n-1}^i - \bar{f}_{n-1}^r + \bar{I}\bar{f}_{n-1}^t \qquad ...(5.8)$$

where

$$\bar{I} = \bar{\rho}_a\bar{c}_a/\bar{\rho}_b\bar{c}_b,\ f_{n-1}^j \equiv c\bar{A}_{n-1}^j,\ j = i, r, t. \qquad ...(5.9)$$

Solving eqns. (5.7) and (5.8) we have

$$\bar{A}_n^r = [(1 - \bar{I})\bar{A}_n^i - \bar{f}_{n-1}^i - \bar{f}_{n-1}^r + \bar{I}\bar{f}_{n-1}^t](1 + \bar{I})^{-1} \qquad ...(5.10)$$

and

$$\bar{A}^t_n = [2\bar{A}^i_n - \bar{f}^i_{n-1} - \bar{f}^r_{n-1} + \bar{I}\bar{f}^t_{n-1}]\,(1 + \bar{I})^{-1}. \qquad ...(5.11)$$

Thus, since $\bar{A}^i_n$ and $\bar{h}^i$ are known, the initial values of $\bar{A}^r_n$, $\bar{A}^t_n$, $\bar{h}^r$ and $\bar{h}^t$ at the interface are known, and the procedures of the preceding section may be applied to obtain the formal solution for the reflected and transmitted waves.

It can be easily noticed from (5.10) that in the limiting case of $\bar{I} = 1$ (that is, $\bar{\rho}_a\bar{c}_a = \bar{\rho}_b\bar{c}_b$), there is no reflected wave only if $\bar{f}^i_{n-1} = \bar{f}^t_{n-1} - \bar{f}^r_{n-1}$ for all $n \geqslant 1$. Thus, even though the impedance is continuous at $x = \bar{x}$ it is still possible that a wave will be reflected. Also it should be noted that, in the case $\bar{I} = 1$, the reflected wave has no discontinuity since $\bar{A}^r_0 = 0$.

The asymptotic methods empolyed in this paper can similarly be used to treat multiple interfaces.

References

Achenbach, J. D., and Reddy, D. P. (1967). Note on wave propagation in linear viscoelastic media. *ZAMP*, **18**, 141–44.

Bland, D. R. (1960). The Theory of Linear Viscoelasticity. Pergamon Press, Inc., New York.

Christensen, R. M. (1971). Theory of Viscoelasticity, An Introduction. Academic Press, New York.

Friedlander, F. G. (1947). Simple progressive solutions of the wave equation. *Proc. Camb. phil. Soc.*, **43**, 360–73.

Karal, F. C. (Jr.), and Keller, J. B. (1959). Elastic waves propagation in homogeneous and inhomogeneous media. *J. acoust. Soc. Am.*, **31**, 694–705.

Kolsky, H. (1963). Stress Waves in Solids. Dover Publs., New York.

Moodie, T. B. (1973). On the propagation, reflection and transmission of transient cylindrical shear waves in nonhomogeneous four-parameter viscoelastic media. *Bull. Austr. math. Soc.*, **8**, 397–411.

Park, I. K., and Reiss, E. L. (1970). Oscillatory impact of an inhomogeneous viscoelastic rod. *J. acoust. Soc. Am.*, **47**, 870–74.