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ON TOTALLY REAL SUBMANIFOLDS OF A KÄHLERIAN MANIFOLD ADMITTING THE COMPLEX CONFORMAL CONNECTION

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A Kählerian manifold admitting a complex conformal connection with vanishing Ricci tensor is considered. It is shown that the Bochner curvature tensor is identically equal to the complex conformal curvature tensor relative to the complex conformal connection and generalise the result of Yano (1975). The totally real submanifold of a Kählerian manifold is also considered and it is found that the induced connection on a totally real submanifold of a Kählerian manifold with the complex conformal connection is the conformal connection. Finally, the umbilical properties of totally real submanifolds are studied.

1. INTRODUCTION

Yano (1975) has defined the complex conformal connection and proved that if a Kähler manifold admits the complex conformal connection with vanishing curvature tensor, the Bochner curvature tensor of the manifold vanishes. In this paper we show that a Kählerian manifold admitting the complex conformal connection with vanishing Ricci tensor the Bochner curvature tensor is identically equal to the complex conformal curvature tensor relative to the complex conformal connection. Thus above mentioned result of Yano follows as a consequence of this result.

Next we consider the totally real submanifold of a Kählerian manifold and show that if a Kählerian manifold admits the conformal change of Hermitian metric, the totally real submanifold admits a conformal change of induced Riemannian metric. We also show that the induced connection on a totally real submanifold of a Kählerian manifold with the complex conformal connection is the conformal connection. Finally, we study the umbilical properties of the totally real submanifolds and obtain the results analogous to that of Yano (1976).

2. PRELIMINARIES

Let M^{2m} $(m > 2)$ be a Kählerian manifold covered by a system of coordinate neighbourhoods (U, x^h), where the indices $h, i, j, \ldots$ run over the range $\{1, 2, \ldots, 2m\}$ and let g_{ji}, $F_i^{\ h}$, ∇_h, $K_{kji}^{\ \ \ h}$, K_{ji} and K denote the components of Hermitian metric tensor, the complex structure tensor, the operator of covariant differentiation with respect to Christoffel symbols $\left\{ {h \atop ij} \right\}$, the curvature tensor, the Ricci tensor and the scalar curvature of M^{2m}.

Suppose the Kählerian manifold M^{2m} admits the complex conformal connection ∇^*_i (Yano 1975) with connection coefficients Γ^{*h}_{ji} given by

$$\Gamma^{*h}_{ji} = \left\{ {h \atop ji} \right\} + \delta^h_j p_i + \delta^h_i p_j - g_{ji} p^h + F^h_j q_i + F^h_i q_j - F_{ji} q^h \qquad ...(2.1)$$

where

$$p_i = \partial_i p, \quad \partial_i = \frac{\partial}{\partial x^i}, \; p^h = p_t g^{th}, \quad q_i = -p_t F^t_i \text{ and } q^h = q_t g^{th},$$

p being a scalar function. The curvature tensor K^{*h}_{kji} of Γ^{*h}_{ji} and K^h_{kji} of $\left\{ {h \atop ji} \right\}$ are related by (Yano 1975)

$$\begin{aligned} K^{*h}_{kji} = K^h_{kji} &- \delta^h_k p_{ji} + \delta^h_j p_{ki} - p^h_k g_{ji} + p^h_j g_{ki} \\ &- F^h_k g_{ji} + F^h_j q_{ki} - q^h_k F_{ji} + q^h_j F_{ki} \\ &+ (\nabla_k q_j - \nabla_j q_k) F^h_i - 2F_{kj}(p_i q^h - q_i p^h) \end{aligned} \qquad ...(2.2)$$

where

$$p_{ji} = \nabla_j p_i - p_j p_i + q_j q_i + \tfrac{1}{2} p^t p_t g_{ji} \qquad ...(2.3)$$

$$q_{ji} = \nabla_j q_i - p_j q_i - q_j p_i + \tfrac{1}{2} p_t p^t F_{ji} \qquad ...(2.4)$$

consequently from (2.3) and (2.4)

$$q_{ji} = -p_{jt} F^t_i, \quad p_{ji} = q_{jt} F^t_i$$

$$p^h_k = p_{kt} g^{th}, \quad q^h_k = q_{kt} g^{th}.$$

we set

$$\alpha_{kj} = -(\nabla_k q_j - \nabla_j q_k) \qquad ...(2.5)$$

and

$$\beta_{kj} = 2(p_t q^h - q_t p^h). \qquad ...(2.6)$$

(2.5) and (2.6) can also be written as [see for computation Yano (1975),

$$\alpha_{ji} = -2q_{ji} - \frac{2}{n+4} p^t_t F_{ji} \qquad ...(2.7)$$

and

$$\beta_{ji} = -2q_{ji} + \frac{2}{n+4} p_t^{\ t} F_{ji} \quad ..\ (2.8)$$

where

$$p_t^{\ t} = \nabla_t p^t + \frac{n}{2} p^t p_t.$$

From (2.2), we have

$$K^*_{kjih} = K_{kjih} - g_{kj} p_{ji} + g_{jh} p_{ki} - p_{kh} g_{ji} + p_{jh} g_{ki}$$
$$- F_{kh} q_{ji} + F_{jh} q_{ki} - q_{kh} F_{ji} + q_{jh} F_{ki} - \alpha_{kj} F_{ih} - F_{kj} \beta_{ih} \quad ...(2.9)$$

where

$$K^*_{kjih} = K^{*\ t}_{kji}\, g_{th},\ K_{kjih} = K_{kji}^{\ \ \ t}\, g_{th}.$$

Transvecting (2.9) by g^{kh} and after a long computation (Yano 1975), we find

$$K^*_{ji} = K_{ji} - 2(m+2)\, p_{ji} - p_t^{\ t}\, g_{ji} \quad ...(2.10)$$

where K^*_{ji} being the Ricci tensor with respect to ∇^*_i. From (2.10) again transvecting by g^{ji},

$$K^* = K - 4(m+1)\, p_t^{\ t} \quad ...(2.11)$$

where K^*, the scalar curvature with respect to ∇^*_i.

We now consider the Bochner curvature tensor (Yano and Bochner 1953) defined by

$$B_{kji}^{\ \ \ h} = K_{kji}^{\ \ \ h} + \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_k^{\ h} h_{ji} - L_j^{\ h} g_{ki}$$
$$+ F_k^{\ h} M_{ji} - F_j^{\ h} M_{ki} + M_k^{\ h} F_{ji} - M_j^{\ h} F_{ki}$$
$$- 2(M_{ki} F_i^{\ h} + F_{kj} M_i^{\ h}) \quad ...(2.12)$$

where

$$\left.\begin{aligned} L_{ji} &= -\frac{1}{2m+4} K_{ji} + \frac{1}{2(2m+2)(2m+4)} K g_{ji} \\ M_{ji} &= -L_{jt} F_i^{\ t} \end{aligned}\right\} \quad ...(2.13)$$

that is

$$\left. \begin{aligned} M_{ji} &= -\frac{1}{2(m+2)} H_{ji} + \frac{1}{2(2m+2)(2m+4)} KF_{ji} \\ \text{and} \quad & \\ L^h_k &= L_{kt}g^{th},\ M^h_k = M_{kt}g^{th},\ H_{ji} = -K_{jt}F^t_i . \end{aligned} \right\} \quad \ldots(2.14)$$

From (2.12), we have

$$\begin{aligned} B_{kjih} = K_{kjih} &+ g_{kh}L_{ji} - g_{jh}L_{ki} + L_{kh}g_{ji} - L_{jh}g_{ki} \\ &+ F_{kh}M_{ji} - F_{jh}M_{ki} + M_{kh}F_{ji} - M_{jh}F_{ki} \\ &- 2(M_{kj}F_{ih} + F_{kj}M_{ih}) \end{aligned} \quad \ldots(2.15)$$

where $B_{kjih} = B^t_{kji}\, g_{th}$.

Let M^n be a totally real submanifold of a Kählerian manifold M^{2m} $(m > 2)$ with induced metric g_{ab} and the induced Christoffel symbol $\left\{ {a \atop bc} \right\}$ and let ∇_b, K^a_{dcb}, K_{ba} and R the operator of covariant differentiation with respect to $\left\{ {a \atop bc} \right\}$, the curvature tensor, the Ricci tensor and the scalar curvature of M^n.

We set

$$\Gamma^{*a}_{bc} = (\partial_c B^h_b + \Gamma^{*h}_{ji} B^j_b B^i_c)\, B^a_h \quad \ldots(2.16)$$

where

$$B^h_c = \frac{x^h}{u^c} \quad \text{and} \quad B^a_h = g^{ab}g_{hi}B^i_b .$$

Then Γ^{*a}_{bc} is the induced connection on M^n with the induced metric g^*_{bc}. We denote $\triangle^*_b$, the operator of covariant differentiation with respect to Γ^{*a}_{bc} and K^{*a}_{dcb}, K^*_{bc} and K^* the curvature tensor, the Ricci tensor and the scalar curvature of M^n with respect to ∇^*_b. We put

$$\nabla^*_c B^h_b = \partial_c B^h_b + \Gamma^{*h}_{ji} B^j_c B^i_b - \Gamma^{*a}_{cb} B^h_a,\ \partial_c = \frac{\partial}{\partial u^c} \quad \ldots(2.17)$$

where Γ^{*h}_{ji} and Γ^{*a}_{bc} are given by (2.1) and (2.16). We call this kind of covariant differentiation the Van-der Waerden-Bartolotti covariant differentiation with respect to the complex conformal connection.

Suppose $C_1^h, \ldots, C_{2m-n}^h$ are $2m - n$ unit orthogonal normal fields on M^n. Decomposing p^h into its unique tangential and normal components along M^n, we get

$$p^h = p^a B_a^h + \alpha^x C_x^h \qquad \ldots(2.18)$$

where the summation in the index x runs over the range $x = 1, 2, \ldots, 2m - n$ and $p^a = g^{ab} p_b$, $p_b = \nabla_b p = \partial_b p$.

Define the second fundamental tensor H_{cb}^{*x} of ∇_c^* relative to the normal C_x^h by

$$H_{cbx}^* = g_{jh}(\nabla_c^* B_b^j)\, C_x^h. \qquad \ldots(2.19)$$

The Gauss curvature equation of M^n with respect to the complex conformal connection is given by

$$K_{kjih}^*\, B_d^k\, B_c^j\, B_b^i\, B_a^h = K_{dcba}^* - A_{dcba}^* \qquad \ldots(2.20)$$

where

$$A_{dcba}^* = H_{da}^{*x}\, H_{cbx}^* - H_{ca}^{*x}\, H_{dbx}^*. \qquad \ldots(2.21)$$

The well-known Weyl's conformal curvature tensor C_{dcba} of M^n is given by

$$C_{dcba} = K_{dcba} - \frac{1}{n-a}(g_{da}K_{cb} - g_{db}K_{ca} + g_{cb}K_{da} - g_{ca}K_{db})$$

$$+ \frac{R}{(n-1)(n-2)}(g_{da}g_{cb} - g_{db}g_{ca}) \qquad \ldots(2.22)$$

for $n > 3$.

3. Complex Conformal Curvature Tensor Relative to the Complex Conformal Connection with Vanishing Ricci Tensor

In this section we prove the following theorem:

Theorem 3.1 — Let M^{2n} $(m > 2)$ be a Kählerian manifold admitting the complex conformal connection (2.1). If the Ricci tensor with respect to the complex conformal connection vanishes, the Bochner curvature tensor is identically equal to the curvature tensor of the complex conformal connection.

PROOF : If $K_{ji}^* = 0$, from (2.10) and (2.11), we have

$$K_{ji} = 2(m+2)\, p_{ji} + p_t^t\, g_{ji} \qquad \ldots(3.1)$$

and

$$K = 4(m+1)\, p^{t}_{t}\,. \qquad ...(3.2)$$

Substituting (3.1) and (3.2) into (2.13) and (2.14) we find (Yano 1975)

$$L_{ji} = -p_{ji}, \quad M_{ji} = -q_{ji}. \qquad ...(3.3)$$

Now substituting (3.3) into (2.15), we find

$$B_{kjih} = K_{kjih} - g_{kh}p_{ji} + g_{jh}p_{ki} - p_{kh}g_{ji} + p_{jh}g_{ki} - F_{kh}q_{ji} + F_{jh}q_{ki} - q_{kh}F_{ji} + q_{jh}F_{ki} + 2(q_{kj}F_{ih} + F_{kj}q_{ih}). \qquad ...(3.4)$$

From (2.7) and (2.8), we have

$$\alpha_{kj}F_{ih} + F_{kj}\beta_{ih} = -2(q_{kj}F_{ih} + F_{kj}q_{ih}).$$

With this eqn. (3.4) becomes

$$B_{kjih} = K^{*}_{kjih}$$

where we have used (2.9). This proves Theorem 3.1.

As a consequence of this theorem, we have the known corollary.

Corollary 3.1 (*Yano* 1975) — If in a Kähler manifold M^{2m} a scalar function p is such that the complex conformal connection (2.1) is of zero curvature, the Bochner curvature tensor of the manifold vanishes.

PROOF : If $K^{*}_{kjih} = 0$, $K^{*}_{ji} = 0$. Thus from Theorem 3.1 $B_{kjih} = K^{*}_{kjih}$. As $K^{*}_{kjih} = 0$, the Bochner curvature tensor of the manifold vanishes. Thus we have the proof of the corollary.

4. Conformally Flat Totally Real Submanifolds of a Kählerian Manifold

In this section, we study the umbilical properties of the submanifolds.

Theorem 4.1 — If a Kählerian manifold M^{2m} $(m > 2)$ admits a conformal change of Hermitian metric, the totally real submanifold M^{n} admits a conformal change of a Riemannian metric.

PROOF : Suppose M^{2m} admits a conformal change of a Hermitian metric, that is

$$g^{*}_{ji} = e^{2p}g_{ji}, \quad F^{*h}_{i} = F^{h}_{i}, \quad F^{*}_{ji} = e^{2p}F_{ji} \qquad ...(4.1)$$

p being a scalar function.

Multiplying both sides of (4.1) by $B^j_b\, B^i_a$ we get

$$g^*_{ba} = e^{2p} g_{ba} \text{ and } F^*_{ji}\, B^j_b\, B^i_a = 0 \qquad ...(4.2)$$

where the induced metric g^*_{ba} is given by $g^*_{ba} = g^*_{ji}\, B^j_b\, B^i_a$ and $F_{ji} B^j_b\, B^i_a = 0$. Thus eqn. (4.2) proves the Theorem.

Remark 4.1 : Note that we have also proved in this theorem that if M^n is a totally real submanifold with induced metric g_{ab}, it is so with respect to the induced metric g^*_{ab} given by (4.2).

Theorem 4.2 — The induced connection (2.16) on a totally real submanifold of a Kählerian manifold with the complex conformal connection (2.1) is the conformal connection formed with g^*_{ab}.

Proof : From (2.16),

$$\Gamma^{*a}_{bc} = (\nabla_c B^h_b + \Gamma^{*h}_{ji}\, B^j_b\, B^i_c)\, B^a_h.$$

Now substituting in this equation for Γ^{*h}_{ji} from (2.1) and using the fact that $F_{ji} B^j_b\, B^i_c = 0$, we get

$$\Gamma^{*a}_{bc} = \begin{Bmatrix} a \\ bc \end{Bmatrix} + \delta^a_b\, p_c + \delta^a_c\, p_b = g_{ac} p^a \qquad ...(4.3)$$

where $p_c = B^h_c\, p_h$, $p^a = g^{ab} p_b$ and $\begin{Bmatrix} a \\ bc \end{Bmatrix}$ is the induced connection on M^n with g_{ab}. Thus from (4.3), the induced connection Γ^{*a}_{bc} is the conformal connection formed with g^*_{bc} given by (4.2).

Lemma 4.1 — Let M^n be a totally real submanifold of a Kählerian manifold M^{2m}. The second fundamental tensor H^{*x}_{bc} of M^n with respect to ∇^*_c is related to the second fundamental tensor of M^n with respect to the Riemannian connection on M^n by the eqation

$$H^{*x}_{cb} = H^x_{cb} - \alpha^x (g_{cb} - L_{cb}) \qquad ...(4.4)$$

where L_{cb} is given by (4.7).

PROOF : From (2.19),

$$H^*_{cbx} = g_{hl}(\nabla^*_c B^h_b)\, C^l_x .$$

Now substituting in this equation for $\nabla^*_c B^h_b$ from (2.17) and further substituting for Γ^{*h}_{ji} and Γ^{*a}_{bc} from (2.1) and (4.3) in the resulting equation, we get

$$H^*_{cbx} = H_{cbx} - g_{cb} p^h C^l_x g_{hl} + F_{jl} F_{ti} p^t C^l_x B^j_c C^i_b + F_{il} F_{tj} p^t C^l_x B^i_c C^j_b \qquad \text{...(4.5)}$$

Now from (2.18) eqn. (4.5) becomes

$$H^*_{cbx} = H_{cbx} - \alpha_x g_{cb} + F_{il} F_{tj} \alpha^y C^t_y B^j_c B^i_b C^l_x + F_{jl} F_{ti} \alpha^y C^t_y B^i_c B^j_b C^l_x \qquad \text{...(4.6)}$$

where we have used $F_{ji} B^j_b B^i_c = 0$. If we set

$$\alpha_x L_{cb} = F_{il} F_{tj} \alpha^y C^t_y B^j_c B^i_b C^l_x + F_{il} F_{ti} \alpha^y C^t_y B^i_c B^j_b C^l_x \qquad \text{...(4.7)}$$

we get (4.4) from (4.6).

The Gauss equation of M^n is given by

$$K_{kjih} B^k_d B^j_c B^i_b B^h_a = K_{dcba} - A_{dcba} \qquad \text{...(4.8)}$$

where

$$A_{dcba} = H^x_{da} H_{cbx} - H^x_{ca} H_{dbx}. \qquad \text{...(4.9)}$$

From (4.9) we have

$$A_{cb} = g^{da} A_{dcba} = H^a_{ax} H^x_{cb} - H^x_{cb} H_{dbx} \qquad \text{...(4.10)}$$

and

$$A = g^{cb} A_{cb} = n^2 H^2 - S^2 \qquad \text{...(4.11)}$$

H and S being lengths respectively of mean curvature vector and the second fundamental tensor.

Similarly from (2.21), we find

$$A^*_{cb} = g^{*cb} A^*_{dcba} = H^{*a}_{ax} H^{*x}_{cb} - H^{*dx}_c H^*_{dbx} \qquad \text{...(4.12)}$$

and

$$A^* = g^{cb} A^*_{cb} = n^2 H^{*2} - S^{*2} \qquad \text{...(4.13)}$$

H^* and S^* being lengths respectively of mean curvature vector

$$H^{*h} = \frac{1}{n} g^{*cb} H^{*x}_{cb} C^h_x$$

and the second fundamental tensor with respect to ∇^*_c. We define the tensor field H_{dcba} of type (0, 4) by

$$H_{dcba} = A_{dcba} = \frac{1}{n-1} (g_{da}A_{cb} - g_{db}A_{ca} + g_{cb}A_{da} - g_{ca}A_{db}) + \frac{A}{(n-1)(n-2)} (g_{da}g_{cb} - g_{db}g_{ca}) \quad ...(4.14)$$

where A_{dcba}, A_{cb} and A are given by (4.9), (4.10) and (4.11) respectively.

Lemma 4.2 — Let M^n $(n > 3)$ be a totally real submanifold of a Kählerian manifold M^{2m} $(m > 2)$.

Then

$$H^*_{dcba} = H_{dcba} \quad ...(4.15)$$

where H_{dcba} is given by (4.14) and H^*_{dcba} is defined similar to that of (4.14)

PROOF : From (2.20), we write

$$K^*_{dcba} = A^*_{dcba} + K^*_{kjih} B^k_d B^j_c B^i_b B^h_a B.$$

Substituting (2.9) in this equation for K^*_{kjih}, we obtain

$$K^*_{dcba} = A^*_{dcba} + K_{kjih}B^k_d B^j_c B^i_b B^h_a - g_{da}Q_{cb} + g_{ca}Q_{db} - g_{cb}Q_{da} + Q_{ca}g_{db} \quad ...(4.16)$$

where

$$Q_{cb} = p_{ji}B^j_c B^i_b .$$

Now using (4.8) in (4.16), we find

$$K^*_{dcba} = A^*_{dcba} + K_{dcba} - g_{da}Q_{cb} + g_{ca}Q_{db} - g_{cb}Q_{da} + Q_{ca}g_{db}. \quad ...(4.17)$$

Contracting (4.17) with respect to the indices d and a

$$K^*_{cb} = A^*_{cb} + K_{cb} - A_{cb} - (n-2)\, Q_{cb} - Qg_{cb} \quad ...(4.18)$$

where $Q = g^{cb}Q_{cb}$ and A^*_{cb} is given by (4.12).

Further transvecting by g^{*cb}, we get

$$e^{2p}K^* = e^{2p}A^* + K - A - 2(n-1)\,Q \qquad \ldots(4.19)$$

where A^* is given by (4.13). We define C^*_{dcba} similar to that of (2.22) by

$$C^*_{dcba} = K^*_{dcba} - \frac{1}{n-1}\,(g^*_{da}\,K^*_{cb} - g^*_{db}\,K^*_{ca} + g^*_{cb}\,K^*_{da} + g^*_{ca}\,K^*_{db}) + \frac{K^*}{(n-1)(n-2)}\,(g^*_{da}\,g^*_{cb} - g^*_{db}\,g^*_{ca}). \qquad \ldots(4.20)$$

Substituting for K^*_{dcba}, K^*_{bc} and K^* from (4.17), (4.18) and (4.19) into (4.20) and using (4.2), we get

$$C^*_{dcba} = C_{dcba} + H^*_{dcba} - H_{dcba}.$$

From which using the fact that $C^*_{dcba} = C_{dcba}$, we get (4.15).

Lemma 4.3 — Let M^n $(n > 3)$ be a totally real submanifold of Kählerian manifold M^{2m} $(m > 2)$. If there exists a scalar function p such that

$$K^*_{kjih} = \alpha_{kj}F_{ih} \qquad \ldots(4.21)$$

or

$$K^*_{kjih} = \beta_{kj}F_{ih} \qquad \ldots(4.22)$$

$$C_{dcba} = C^*_{dcba} = H^*_{dcba} \qquad \ldots(4.23)$$

PROOF : From (4.21) or (4.22), eqn. (2.20) becomes

$$K^*_{dcba} = A^*_{dcba}.$$

Consequently substituting this into (4.20), we get (4.23).

Theorem 4.3 — Under the hypothesis of Lemma 4.3, the totally umbilical, totally real submanifold of a Kählerian manifold is conformally flat.

PROOF : From Lemmas 4.2 and 4.3, we have

$$C_{dcba} = C^*_{dcba} = H^*_{dcba} = H_{dcba}. \qquad \ldots(4.24)$$

If M^n is totally umbilical,

$$H^x_{cb} = H^x g_{cb},\; H^x = \frac{1}{n}\, g^{cb}H^x_{cb}$$

where H^x is the mean curvature with respect to the normal C^h_x. Substituting this into (2.19), we obtain

$$A_{dcba} = (H^x H_x)(g_{da}g_{cb} - g_{ca}g_{db}), \quad A_{cb} = (n-1)(H_x H^x)\, g_{cb}$$

$$A = n(n-1)(H^x H_x).$$

With these equations it is easy to see that,

$$H_{dcba} = 0.$$

So that from (4.24)

$$C_{dcba} = 0.$$

Thus from the well-known theorem of Weyl, M^n is conformally flat.

Remark 4.2 : Yano (1976) has obtained the similar result with the assumption of vanishing Bochner curvature tensor. But here note that neither the condition (4.21) nor (4.22) implies the vanishing of Bochner curvature tensor.

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