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FINSLER SPACES WITH THE BERWALD'S CURVATURE TENSOR OF A SPECIAL FORM

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In this paper a special form of the Berwald's curvature tensor H^i_{jkl} of a Finsler space is proposed. It arises from the theory of Finsler space of non-zero scalar curvature. A Landsberg space of such a special form of H^i_{jkl} is c-reducible while a Berwald space of such form of H^i_{ijk} is Riemannian.

1. PRELIMINARIES

Let F_n $(n \geqslant 3)$ be an n-dimensional Finsler space equipped with metric function $L(x, y)$. The Berwald's connection parameters G^i_{jk} and the Cartan's connection parameters Γ^{*i}_{jk} gives rise to two h-covariant differentiation in F_n (Matsumoto 1970). We shall denote (j) as the Berwald's h-covariant differentiation and $|\,j$ as the Cartan's h-covariant differentiations. We shall also denote $\|\,j$ as the partial differentiation with respect to the element of support y^j. A Finsler space of scalar curvature R is characterized by the relation (Numata 1975)

$$R_{iok} = g_{ip}\, R^p_{ok} = RL^2 h_{ik} \qquad \ldots(1.1)$$

where $h_{ik} = g_{ik} - L^{-2} y_i y_k$ is the angular metric tensor, R^i_{jk} is the (v) h-torsion tensor of F_n and the suffix o means the contraction with the element of support y^i.

If the Cartan's third curvature tensor R_{lijk} of F_n is written in the form

$$R_{lijk} = R(g_{lj} g_{ik} - g_{lk} g_{ij}) \qquad \ldots(1.2)$$

where R is non zero scalar then F_n is said to be h-isotropic (Matsumoto 1971b). If the (h) hv torsion tensor C_{ijk} is written in the form

$$C_{ijk} = \frac{1}{n+1}\,(C_i h_{jk} + C_j h_{ki} + C_k h_{ij}) \qquad \ldots(1.3)$$

where $C_i = C_{ijk} g^{jk}$ is the torsion vector, then F_n is called a c-reducible Finsler space (Matsumoto 1972). A Landsberg space is characterized by the relation $C_{ijk|o} = 0$ which is equivalent to

$$y_p G^p_{ljk} = 0 \qquad \ldots(1.4)$$

where $\quad G^p_{ljk} = G^p_{lj\|k} = G^p_{jk\|l} = G^p_{kl\|j}$.

It is well known that the Berwald's curvature tensor H^i_{ljk} satisfies the identities (Rund 1959)

$$H^i_{ljk} = -H^i_{lkj} \qquad \ldots(1.5)$$

$$H^i_{ojk} = R^i_{ojk} = R^i_{jk}, \quad R^i_{ljk} = g^{ip} R_{lpjk} \qquad \ldots(1.6)$$

$$H^i_{ljk} + Cyl\,(ljk) = 0 \qquad \ldots(1.7)$$

$$H^i_{ljk(m)} + R^p_{jk}\, G^i_{plm} + Cyl\,(jkm) = 0 \qquad \ldots(1.8)$$

where the notation $Cyl\,(ljk)$ denotes the cyclic permutation of indices l, j, k and summation.

Further the Berwald's curvature tensor H^i_{ljk} is related with the (v) h-torsion tensor R^i_{jk}, the Cartan's third curvature tensor R_{lijk} and the tensor G^i_{ljk} as follows

$$H^i_{ljk} = R^i_{jk\|l} \qquad \ldots(1.9)$$

$$\$_{(jk)} \{G^i_{lkm(j)}\} = -H^i_{ljk\|m} \qquad \ldots(1.10)$$

$$R_{lijk} = \$_{(il)} \{\tfrac{1}{2} H_{lijk} - C^p_{lj|o}\, C_{ipk|o}\}, \; H_{lijk} = g_{im} H^m_{ljk} \qquad \ldots(1.11)$$

where $\$_{(jk)}$ denotes the interchange of indices j, k and subtraction.

The angular metric tensors h_{ij} and $h^i_j \,(= g^{ik} h_{kj})$ has the following covariant differentiations

$$h_{ij|k} = h^i_{j|k} = 0 \qquad \ldots(1.12)$$

$$h_{ij(k)} = -2C_{ijk|o}, \quad h^i_{j(k)} = 0 \qquad \ldots(1.13)$$

$$\left.\begin{aligned}&\text{(a)}\quad h_{ij\|k} = 2C_{ijk} - L^{-2}\,(y_i h_{jk} + y_j h_{ik})\\ &\text{(b)}\quad h^i_{j\|k} = -L^{-2}(y_j h^i_k + y^i h_{jk}).\end{aligned}\right\} \quad ...(1.14)$$

We shall use the following lemmas which have been proved by Matsumoto (1978).

Lemma 1 — If the equation $v_i h_{jk} - v_j h_{ik} = 0$ holds in F_n, then we have (1) $v_i = 0$ $(n \geqslant 3)$ and (2) $v_i = \nu m_i$ $(n = 2)$ where ν is a scalar.

Lemma 2 — If the equation $v_{hi} h_{jk} + v_{ij} h_{hk} + v_{jh} h_{ik} = 0$ holds in F_n, then (1) $v_{ij} = 0$ $(n \geqslant 4)$ and (2) $v_{ij} = \nu\,(m_i n_j - m_j n_i)$ with reference to the Moör frame (l^i, m^i, n^i), where ν is a scalar.

2. Special Form of H^i_{ljk}

In the present paper we shall study a Finsler space F_n with Berwald's curvature tensor H^i_{ljk} of the following form:

$$H^i_{ljk} = D_{jk} h^i_l + \$_{(jk)}\,\{A_{jl} h^i_k + B^i_k\; h_{jl}\} \quad ...(2.1)$$

where D_{jk}, A_{jk} and B^i_k are some Finsler tensor fields. As examples of such spaces we have the following.

Proposition 1 — A Finsler space of non-zero scalar curvature has the special form (2.1) of Berwald's curvature tensor H^i_{ljk}.

Proof : Since $R^i_{jk} = \frac{1}{3}\,(R^i_{ok\|j} - R^i_{oj\|k})$, from (1.1) and (1.14) it follows that

$$R^i_{jk} = a_j h^i_k - a_k h^i_j \quad ...(2.2)$$

where $\quad a_j = \frac{1}{3}\,(R_{\|j}\, L^2 + 3R y_j)$.

From (2.2), (1.9) and (1.14*b*) it follows that H^i_{ljk} can be written in the form (2.1) with

$$D_{jk} = L^{-2}\,(a_k y_j - a_j y_k),\; A_{jk} = a_{j\|k},\; B^i_k = L^{-2}\, a_k y^i.$$

Proposition 2 — An *h*-isotropic Finsler space F_n has the special form (2.1) of Berwald's curvature tensor H^i_{ljk}.

Proof : Contracting (1.2) with y^l we have in view of (1.6)

$$R_{ijk} = g_{il}\, R^l_{jk} = \$_{jk}\,\{R y_j h_{ik}\}. \quad ...(2.3)$$

The relation (2.3), (1.9) and (1.14) gives the special form (2.1) of $H^{\,i}_{ljk}$ where

$$D_{jk} = 0,\ A_{jk} = Rg_{jk} + y_j R_{\|k},\ B^{\,i}_{k} = RL^{-2}\, y_k y^i.$$

We now consider a Finsler space F_n $(n \geqslant 3)$ with $H^{\,i}_{ljk}$ of the form (2.1). From (1.5) and (2.1) it follows that

$$D_{jk} + D_{kj} = 0. \qquad \text{...(2.4)}$$

Next from (1.6) and (2.1), we obtain

$$R^{\,i}_{jk} = A_{jo}\, h^{\,i}_{k} - A_{ko}\, h^{\,i}_{j}\,. \qquad \text{...(2.5)}$$

Contracting (2.5) with y^j we get

$$R^{\,i}_{ok} = A_{oo}\, h^{\,i}_{k}\,, \qquad \text{...(2.5a)}$$

Comparing equations (2.5*a*) and (1.1) we have the following

Proposition 3 — If the Berwald's curvature $H^{\,i}_{ljk}$ of F_n is of the form (2.1), then F_n is a Finsler space of a scalar curvature $L^{-2}A_{oo}$.

Contracting (2.5*a*) and (2.5) with respect to the indices i and k we get

$$H = \frac{1}{n-1}\, R^{\,i}_{oi} = A_{oo},\ H_j = R^{\,i}_{ji} = (n-2)\, A_{jo} + L^{-2} A_{oo} y_j. \qquad \text{...(2.6)}$$

Hence we have

$$A_{jo} = \frac{1}{n-2}\,(H_j - L^{-2} H y_j). \qquad \text{...(2.7)}$$

Next we shall deal with (1.7). By (2.1) it can be written as

$$E_{lj}\, h^{\,i}_{k} + E_{kl}\, h^{\,i}_{j} + E_{jk}\, h^{\,i}_{l} = 0 \qquad \text{...(2.8)}$$

where we put $E_{lj} = D_{lj} - (A_{lj} - A_{jl})$.

Applying lemma 2 we get

$$D_{lj} = A_{lj} - A_{jl},\ n \geqslant 4 \qquad \text{...(2.9)}$$

$$D_{lj} = A_{lj} - A_{jl} + \alpha(m_l n_j - m_j n_l),\ n = 3 \qquad \text{...(2.10)}$$

where α is a scalar.

Now we shall treat (1.8). In view of (2.1) and (1.13) we get

$$H^{\,i}_{ljk(m)} = \$_{(jk)}\, \{A_{jl(m)}\, h^{\,i}_{k} + B^{\,i}_{k(m)}\, h_{jl} - 2B^{\,i}_{k}\, C_{jlm\,|\,o}\} + D_{jk(m)}\, h^{\,i}_{l}\,. \qquad \text{...(2.11)}$$

On the other hand from (2.5) and the relation $G^{\,i}_{plm}\, y^p = 0$ we get

$$H^{p}_{jk}\, G^{i}_{plm} = A_{jo} G^{i}_{klm} - A_{ko} G^{i}_{jlm}. \qquad \text{...(2.12)}$$

In view of (2.11) and (2.12) the identity (1.8) gives

$$E_{jlm}\, h^{i}_{k} + D_{jk(m)}\, h^{i}_{l} + F^{i}_{km}\, h_{jl} + Cyl\,(jkm) = 0 \qquad \text{...(2.13)}$$

where

$$E_{jlm} = A_{jl(m)} - A_{ml(j)}, \quad F^{i}_{km} = B^{i}_{k(m)} - B^{i}_{m(k)}.$$

Contracting (2.13) with y_i we get

$$F_{km} h_{jl} + F_{mj} h_{kl} + F_{jk} h_{ml} = 0 \qquad \text{...(2.14)}$$

where

$$F_{km} = B_{ko(m)} - B_{mo(k)},\; B_{kj} = B^{i}_{k}\, g_{ij}.$$

Applying lemma 2 to (2.14) we get

$$B_{ko(m)} = B_{mo(k)} \text{ for } n \geqslant 4 \qquad \text{...(2.15)}$$

$$B_{ko(m)} - B_{mo(k)} = \lambda(m_k n_m - m_m n_k) \text{ for } n = 3 \qquad \text{...(2.16)}$$

where λ is a scalar.

On the other hand contracting the indices i and l in (2.13) and using (2.9), (2.15) we get

$$nD_{jk(m)} + N_{kj(m)} + L^{-2} T_{jk} y_m + Cyl\,(jkm) = 0 \qquad \text{...(2.17)}$$

where

$$N_{jk} = B_{jk} - B_{kj},\; T_{jk} = A_{jo(k)} - A_{ko(j)}. \qquad \text{...(2.18)}$$

Finally we shall treat (1.9). From (1.9) and (2.5) we get

$$H^{i}_{ljk} = L^{-2}(A_{ko} y_j - A_{jo} y_k)\, h^{i}_{l} + \$_{jk}\, \{A_{jo\|l}\, h^{i}_{k} + L^{-2} A_{ko} y^{i} h_{jl}\}. \qquad \text{...(2.19)}$$

This relation and Bianchi identity (1.7) gives

$$M_{jl} h^{i}_{k} + M_{lk} h^{i}_{j} + M_{kj} h^{i}_{j} = 0 \qquad \text{...(2.20)}$$

where $\quad M_{jl} = A_{jo\|l} - A_{lo\|j} + L^{-2}(A_{jo} y_l - A_{lo} y_j).$

Applying Lemma 2 to (2.20) we get

$$A_{jo\|l} - A_{lo\|j} = L^{-2}(A_{lo} y_j - A_{jo} y_l) \text{ for } n \geqslant 4. \qquad \text{...(2.21)}$$

By (2.1) and (2.19) we get

$$\{D_{jk} - L^{-2}(A_{ko}y_j - A_{jo}y_k)\} h^i_l + \$_{jk}\{(A_{jl} - A_{jo||l}) h^i_k$$

$$+ (B^i_k - L^{-2}A_{ko}y^i) h_{jl}\} = 0. \qquad ...(2.22)$$

Contracting (2.22) with y_i we get

$$F_k h_{jl} - F_j h_{kl} = 0 \qquad ...(2.23)$$

where

$$F_k = B_{ko} - A_{ko}.$$

Applying lemma 1 we get $F_k = 0$ i.e.

$$A_{ko} = B_{ko}. \qquad ...(2.24)$$

The relations (2.15), (2.18) and (2.24) lead to $T_{jk} = 0$. On the other hand contracting (2.22) in the indices i and l and using (2.9), (2.24) and (2.21) we get

$$nD_{jk} + N_{jk} - (n+1)(A_{jo||k} - A_{ko||j}) = 0, \ (n \geqslant 4) \qquad ...(2.25)$$

Summarizing the above results we have the following:

Theorem 1 — If the Berwald's curvature tensor H^i_{ljk} of $F_n (n \geqslant 4)$ is of the form (2.1) then we have

(1) $D_{ij} = A_{ij} - A_{ji}$

(2) $A_{jo} = \frac{1}{n-2}(H_j - L^{-2}Hy_j) = B_{jo}$

(3) $A_{oo} = B_{oo} = H$

(4) $A_{io(j)} = A_{jo(i)}$

(5) $nD_{ij(k)} + N_{ji(k)} + Cyl_{(ijk)} = 0$

(6) $nD_{jk} + N_{jk} + (n+1)(A_{ko||j} - A_{jo||k}) = 0.$

3. Landsberg Space With H^i_{ljk} of the Form (2.1)

Theorem 2 — A Landsberg space with H^i_{ljk} of the form (2.1) is a c-reducible Finsler space.

Proof: It has been proved by Numata (1975) that a Landsberg space of non zero scalar curvature is c-reducible. In view of Proposition 3 a Finsler space with

H^{i}_{ljk} of the form (2.1) is a space of non-zero scalar curvature $L^{-2} A_{oo}$. Hence a Landsberg space with H^{i}_{ljk} of the form (2.1) is c-reducible.

It has been also proved by Numata that a Berwald space of non-zero scalar curvature is a Riemannian space. Thus in view of Proposition 3 we have the following:

Theorem 3 — If the Berwald's curvature tensor H^{i}_{ljk} of a Berwald space is of the form (2.1), then it is Riemannian.

Theorem 4 — If the Berwald's curvature tensor of a c-reducible Finsler space is of the form (2.1), then Cartan's third curvature tensor R_{lijk} is given by

$$R_{lijk} = L_{jl}h_{ik} + L_{ki}h_{jl} - L_{kl}h_{ij} - L_{ji}h_{kl} \qquad \ldots(3.1)$$

where

$$L_{ij} = \tfrac{1}{2}(A_{ij} + B_{ij}) + \frac{1}{(n+1)^2}\left(C_{i\,|\,o}\, C_{j\,|\,o} + \frac{u}{2}\, h_{ij}\right) \qquad \ldots(3.2)$$

and $u = C^{i}_{\ |\,o}\, C_{i\,|\,o}$.

PROOF: The Cartan's h-covariant differentiation of (1.3) and contraction with the element of support yield

$$C_{ijk\,|\,o} = \frac{1}{n+1}\left(C_{i\,|\,o}h_{jk} + C_{j\,|\,o}h_{ki} + C_{k\,|\,o}h_{ij}\right). \qquad \ldots(3.3)$$

Substituting (3.3) and (2.1) in (1.11) we get (3.1).

In view of Theorems 2 and 4 we have the following:

Corollary — If the Berwald's curvature tensor H^{i}_{ljk} of a Landsberg Finsler space is of the form (2.1), then the Cartan's third curvature tensor is of the form (3.1).

If $F_n(n \geqslant 3)$ is a Landsberg space then from (1.14) we have $y_i \,.\, G^{i}_{ljk(o)} = 0$. Transvecting (1.10) with $y^k y_i$ and using the fact that $G^{i}_{lkm(j)}\, y^k = 0$ we get

$$H^{i}_{ljk\|m}\, y^k y_i = 0. \qquad \ldots(3.4)$$

In view of (3.4), (1.14) and (2.1) we have

$$C_{jlm} = \lambda_m h_{jl} + \mu_j h_{lm} + \nu_l h_{jm} \qquad \ldots(3.5)$$

where

$$\left.\begin{aligned} \lambda_m &= -\frac{1}{2B_{oo}} B^i_{k\|m}\, y^k y_i \\ \mu_j &= \frac{1}{2B_{oo}} (D_{jo} + L^{-2}B_{oo}y_j - B_{jo}) \\ \nu_l &= \frac{1}{2B_{oo}} (L^{-2}B_{oo}y_l - A_{ol}). \end{aligned}\right\} \quad ...(3.6)$$

Since C_{jlm} is symmetric in the indices j, l and m we have in view of Lemma 1, $\lambda_i = \mu_i = \nu_i$ which shows that F_n is c-reducible. This is the alternative proof of Theorem 2. Substituting $\lambda_i = \mu_i = \nu_i$ in (3.5) and contracting with g^{lm} we get $\lambda_i = \frac{1}{n+1} C_i$. Hence from (3.6) we have the following:

Theorem 5 — If the Berwald's curvature tensor of a Landsberg Finsler space F_n ($n \geqslant 3$) is of the form (2.1), then

(1) $D_{jo} = B_{jo} - A_{oj}$.

(2) $2B_{oo}C_j + (n+1)\, B^i_{k\|j}\, y^k y_i = 0$

(3) $2B_{oo}C_j = (n+1)\,(L^{-2}B_{oo}y_j - A_{oj})$.

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