Indian J. pure appl. Math., **11**(10) : 1278-1285, October 1980

HYPERSURFACES OF *C*-REDUCIBLE FINSLER SPACES

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(*Received* 7 *January* 1980)

The notion of *C*-reducible Finsler spaces has been introduced by Matsumoto (1972). The object of this paper is to study the properties of hypersurfaces immersed in *C*-reducible Finsler space. It has been proved that each hypersurface of a *C*-reducible Finsler space is *C*-reducible. The conditions under which a hypersurface of a *C*-reducible Landsberg space will be Landsberg have been obtained. Finally after using the so-called "*T*-conditions" (Matsumoto 1974) we have explored the situation under which a hypersurface, of a *C*-reducible Finsler space F_n satisfying *T*-conditions, will also satisfy *T*-condition. It has been proved in this context that if both F_n and F_{n-1} satisfy *T*-conditions and F_n is Landsberg then F_{n-1} is also a Landsberg space.

1. INTRODUCTION

Let F_n be a Finsler space of dimensions n with the fundamental function $F(x, \dot{x})$. The components, h_{ij} of angular tensor are given by

$$h_{ij}=g_{ij}-l_i l_j \qquad \ldots(1.1)$$

where g_{ij} are the components of the metric tensor and $l_i = \dfrac{\partial F}{\partial \dot{x}^i}$. The following are the two well-known properties of Finsler spaces.

(P_1) The Berwald connection parameter G^i_{jk} (Rund 1959, p. 79) is not, in general, independent of direction element $\dot{x}$.

(P_2) $g_{ij(k)} = -2C_{ijk\,|\,o} \neq 0$ in general, where (k) stands for Berwald's process of covariant derivation, $C_{ijk} = \frac{1}{2}\left(\dfrac{\partial g_{ij}(x, \dot{x})}{\partial \dot{x}^k}\right)$ and suffix o stands for transvection with respect to $\dot{x}^i$. This property shows that the Ricci Lemma does not, in general, hold with respect to Berwald's process of covariant differentiation.

A Finsler space in which G^i_{jk} is independent of $\dot{x}$ is called a Berwald space. This space is characterized by the condition $C_{ijk\,|\,h} = 0$ (Rund 1959, p. 81).

A Finsler space in which $g_{ij(k)}=0$ is called a Landsberg space. This space is characterized by $C_{ijk\,|\,o} = 0$.

It is obvious that each Berwald space is a Landsberg space. Further the relation $\Gamma^{*i}_{jk} = G^i_{jk} - C^i_{jk|o}$ (Rund 1959, p. 79), involving Cartan's connection parameters Γ^{*i}_{jk}, proves:

Lemma 1 — In a Landsberg space Cartan's and Berwald's connection parameters are identical and in Berwald's space the Cartan's connection parameter is independent of $\dot{x}$.

We shall now define a class of non-Riemannian Finsler spaces in which the conditions characterizing the Landsberg and Berwald spaces are equivalent.

Definition — A Finsler space F_n is called C-reducible (Matsumoto 1972) if

(*i*) F_n is non-Riemannian

(*Ii*) $n \geqslant 3$

(*iii*) The tensor C_{ijk} is given by

$$C_{ijk} = h_{ij}C_k + h_{jk}C_i + h_{ik}C_j \qquad ...(1.2)$$

where

$$C_i = \frac{g^{jk}C_{ijk}}{n+1}.$$

The following lemma is an immediate consequence of eqn. (1.2) and definition of Berwald and Landsberg spaces.

Lemma 2 — The necessary and sufficient condition that a C-reducible Finsler space be a Berwald space (or Landsberg space) is that:

$$C_{i|h} = 0 \text{ (or } C_{i|o} = 0).$$

It has been proved by Matsumoto and Shibata (1959) that each C-reducible Landsberg space is a Berwald space.

2. Hypersurfaces of a C-reducible Finsler Space

Consider a non-Riemannian hypersurface F_{n-1} of $F_n (n \geqslant 4)$, characterized by the equation $x^i = x^i(u^\alpha)$*. The fundamental tensor of F_{n-1} is given by

$$g_{\alpha\beta}(u, \dot{u}) = g_{ij}(x, \dot{x})\, B^i_\alpha\, B^j_\beta \qquad ...(2.1)$$

where $B^i_\alpha = \dfrac{\partial x^i}{\partial u^\alpha}$.

*The indices $i, j, k, \ldots$ take values $1, \ldots, n$ while the indices $\alpha, \beta, \gamma, \ldots,$ assume values $1, 2, \ldots, n-1$.

A calculation based on the well-known relation

$$C_{\alpha\beta\gamma} = C_{ijk}\, B^i_\alpha \;\; B^j_\beta \;\; B^k_\gamma \qquad \text{...(2.2)}$$

equations (1.2), (2.1) and the facts

$$h_{\alpha\beta} = g_{\alpha\beta} - l_\alpha l_\beta, l_\alpha = B^i_\alpha\, l_i \text{ gives}$$

$$C_{\alpha\beta\gamma} = h_{\alpha\beta} C_\gamma + h_{\beta\gamma} C_\alpha + h_{\alpha\gamma} C_\beta \qquad \text{...(2.3)}$$

where

$$C_\alpha = C_i B^i_\alpha = \frac{g^{\beta\gamma} C_{\alpha\beta\gamma}}{n}.$$

This proves the following:

Theorem 2.1 — A hypersurface of a C-reducible Finsler space is a C-reducible Finsler space.

The difference between the intrinsic and induced connection parameters $\hat{\Gamma}^\alpha_{\beta\gamma}$ and $\Gamma^{*\alpha}_{\beta\gamma}$ of a hypersurface has been obtained by Rund (1965). If the space F_n is C-reducible then this difference tensor $\Lambda_{\alpha\gamma\beta} = g_{\gamma\delta}(\hat{\Gamma}^\delta_{\alpha\beta} - \Gamma^{*\delta}_{\alpha\beta})$ reduces to the form:

$$\Lambda_{\alpha\gamma\beta} = \rho\, [h_{\beta\gamma}(\Omega_{\alpha o} - C_\alpha \Omega_{oo}) + h_{\alpha\gamma}(\Omega_{\beta o} - C_\beta\, \Omega_{oo}) - h_{\alpha\beta}(\Omega_{\gamma o} + C_\gamma \Omega_{oo})] \qquad \text{...(2.4)}$$

where $\Omega_{\alpha\beta}$ are the components of the second fundamental tensor of F_{n-1}, $\rho = C_i N^i$ and N^i are the components of the unit vector normal to the hypersurface. This equation proves the following:

Theorem 2.2 — The necessary and sufficient condition that intrinsic and induced connection parameters of a hypersurface of a C-reducible Finsler space be equal is that either $\Omega_{\alpha o} = 0$ or the vector C_i is tangential to the hypersurface.

In order to derive condition under which a hypersurface of a C-reducible Landsberg space be a Landsberg space we note that the induced covariant differentiation of the relation $C_\alpha = B^i_\alpha\, C_i$ yields

$$C_{\alpha\,|\,\beta} = C_{i\,|\,h}\, B^i_\alpha \;\; B^h_\beta + \frac{\partial C_l}{\partial \dot{u}^\alpha} \Omega_{\beta o} N^l + \rho \Omega_{\alpha\beta} \qquad \text{...(2.5)}$$

where we have used the fact that $\dfrac{\partial C_i}{\partial \dot{x}^l}$ is symmetric in the indices i and l.

The transvection of the relation (2.5) with respect to $\dot{u}^\beta$ gives

$$C_{\alpha\,|\,o} = C_{i\,|\,o}\, B^i_\alpha + \frac{\partial C_l}{\partial \dot{u}^\alpha}\, \Omega_{oo}\, N^l + \rho\Omega_{\alpha o}. \qquad ...(2.6)$$

It is now assumed that intrinsic and induced connection parameters are identical then by Theorem 2.2 either $\Omega_{\alpha o} = 0$ or $\rho = 0$.

If $\Omega_{\alpha o} = 0$, then eqn. (2.6) gives

$$C_{\alpha\,|\,o} = C_{i\,|\,o}\, B^i_\alpha\,. \qquad ...(2.7)$$

On the other hand if $\rho = 0$ then eqn. (1.2) shows that the tensor defined by

$$M_{\alpha\beta} = C_{ijk}\, B^i_\alpha\, B^j_\beta\, N^k$$

vanishes. The properties of the hypersurface in this case have been discussed by Brown (1968). He has shown that in this case

$$\frac{\partial N^l}{\partial \dot{u}^\alpha} = -M_\alpha N^l \text{ where } M_\alpha = C_{ijk}\, B^i_\alpha\, N^j N^k.$$

This relation and the condition $\rho = C_l N^l = 0$ give

$$\frac{\partial C_l}{\partial \dot{u}^\alpha} N^l = -C_l \frac{\partial N^l}{\partial \dot{u}^\alpha} = -(C_l N^l)\, M_\alpha = 0.$$

This shows that the condition $\rho = 0$ will again reduce eqn. (2.6) to eqn. (2.7) in which the covariant differentiation on the left side of the equality is intrinsic as well as induced. Therefore the Lemma 2 yields the following:

Theorem 2.3 — If the induced and intrinsic connection parameters of a hypersurface of *C*-reducible Landsberg space are identical, then the hypersurface is Landsberg.

The above discussions show that Theorems 2.2 and 2.3 are valid if any one of the conditions: (i) $\rho = 0$ or (ii) $\Omega_{\alpha o} = 0$ holds. In the following two sections we shall study the properties of the hypersurface in these two cases.

3. The Case $\rho = C_i N^i = 0$

Theorems 2.2 and 2.3 reveal that vanishing of the scalar ρ is only a sufficient condition for the equality of induced and intrinsic connection parameters and also for the fact that the hypersurface of a *C*-reducible Landsberg space is a Landsberg space.

However, the relation (Rund 1959, p. 160)

$$\Gamma^{*\alpha}_{\beta\gamma} = B^\alpha_i \left(\frac{\partial^2 x^i}{\partial u^\beta \partial u^\gamma} + \Gamma^{*i}_{jk}\, B^j_\beta\, B^k_\gamma \right) \qquad ...(3.1)$$

where $B^{\alpha}_{i} = g^{\alpha\beta} . g_{ij} B^{j}_{\beta}$, and the fact that F_n is Landsberg proves that the induced connection parameter $\Gamma^{*\alpha}_{\beta\gamma}$ is independent of direction $\dot{u}$ if and only if B^{α}_{i} is independent of $\dot{u}$. A direct calculation based on the eqns. (1.2), (2.3), (3.1) and the relation $B^{\alpha}_{i} B^{k}_{\alpha} = \delta^{k}_{i} - N^k N_i$ yields

$$\frac{\partial B^{\alpha}_{i}}{\partial \dot{u}^{\gamma}} = \rho g^{\alpha\beta} h_{\beta\gamma} N_i. \qquad ...(3.2)$$

The condition $g^{\alpha\beta} h_{\beta\gamma} = 0$ implies $n = 2$ which contradicts with the assumption $n \geqslant 4$ made earlier. Hence we have the following:

Theorem 3.1 — The necessary and sufficient condition that the induced connection parameter of a hypersurface of a C-reducible Landsberg space be independent of direction element is that the vector field C_i is tangent to the hypersurface i.e. $\rho = 0$.

Theorems 2.2, 2.3 and 3.1 give the following:

Theorem 3.2 — If the induced connection parameter of a hypersurface of a C-reducible Landsberg space be independent of direction element then the induced and intrinsic connection parameters are equal and the hypersurface is a Landsberg space.

4. The Case $\Omega_{\alpha o} = 0$

Theorems 2.2 and 2.3 reveal that the vanishing of $\Omega_{\alpha o}$ is only a sufficient condition for the equality of induced and intrinsic connection parameters and also for the fact that the hypersurface of a C-reducible Landsberg space is a Landsberg space.

In the existing literature two expressions for the normal curvature vector of the hypersurface are given. These two forms of the normal curvature vectors depend on the locally Minkowskian and locally Euclidean characterisations of the Finsler spaces. The two normal curvature vectors denoted by $I^{i}_{\alpha\beta}$, $\overset{\circ}{H}{}^{i}_{\alpha\beta}$ are given by Rund (1959, p. 193) and Davies (1945). These vectors are related by (Rund 1959, p. 193)

$$\overset{\circ}{H}{}^{i}_{\alpha\beta} = I^{i}_{\alpha\beta} + N^i N_j C^{j}_{hk} B^{h}_{\beta} \overset{\circ}{H}{}^{k}_{\alpha\lambda} \dot{u}^{\lambda}. \qquad ...(4.1)$$

The relation immediately shows that

$$\overset{\circ}{H}{}^{i}_{\alpha\beta} \dot{u}^{\beta} = I^{i}_{\alpha\beta} \dot{u}^{\beta} = \Omega_{\alpha o} N^i. \qquad ...(4.2)$$

The eqns. (1.2), (4.1) and (4.2) give

$$\overset{\circ}{H}{}^{i}_{\alpha\beta} = I^{i}_{\alpha\beta} + \Omega_{\alpha o} C_{\beta} N^i.$$

The condition $C_\beta = 0$ implies $C_i = 0$, which shows that the spaces F_n, F_{n-1} are Riemannian. Hence we have the following:

Theorem 4.1 — The necessary and sufficient condition that Rund's and Davies' normal curvature vectors of the hypersurface of a *C*-reducible Finsler space are identical is that $\Omega_{\alpha o} = 0$.

The following two theorems are the immediate consequences of Theorems 2.2, 2.3 and 4.1.

Theorem 4.2 — If Rund's and Davies' normal curvature vectors of the hypersurface of a *C*-reducible Landsberg space are equal then induced and intrinsic connection parameters of the hypersurface are equal.

Theorem 4.3 — If Rund's and Davies' normal curvature vectors of a hypersurface of a *C*-reducible Landsberg space are equal then the hypersurface is also a Landsberg space.

5. *T*-Conditions

We now consider the tensor (Matsumoto (1974) and Kawaguchi (1972)*

$$T_{hijk} = C_{hij} \mid_k + C_{hij} l_k + C_{hik} l_j + C_{hjk} l_i + C_{ijk} l_h. \qquad ...(5.1)$$

The corresponding expression for the tensor $T_{\alpha\beta\gamma\delta}$ can be written down in the space F_{n-1}. The relation

$$C_{\alpha\beta\gamma} = C_{hij} B^h_\alpha B^i_\beta B^j_\gamma$$

yields

$$C_{\alpha\beta\gamma} \mid_\delta = C_{hij} \mid_\delta B^h_\alpha B^i_\beta B^j_\gamma + C_{hij} Z^h_{\alpha\delta} B^i_\beta B^j_\gamma + C_{hij} B^h_\alpha Z^i_{\beta\delta} B^j_\gamma$$
$$+ C_{hij} B^h_\alpha B^i_\beta Z^j_{\gamma\delta} \qquad ...(5.2)$$

where $\qquad Z^h_{\alpha\beta} = B^h_\alpha \mid_\beta.$

Direct calculations will give the relations

$$C_{hij} \mid_\delta = C_{hij} \mid_k B^k_\delta \qquad ...(5.3)$$

*The expression for T_{hijk} is somewhat different from the one given by Matsumoto (1974) in which the first term on the right hand side is $FC_{hij} \mid_k$. This difference arises from the fact that we have used notation given by Rund (1959) while writing down eqn. (5.1). As a matter of fact the term $C_{hij} \mid_k$ occurring in the right hand side of this equation is equal to $FC_{hij} \mid_k$ used by Matsumoto (1974).

and

$$Z^h_{\alpha\delta} = F\rho h_{\alpha\delta} N^h \qquad \text{...(5.4)}$$

where the C-reducibility of the space F_n has been used in proving eqn. (5.4).

In view of eqns. (1.2), (5.3) and (5.4) the relation (5.2) reduces to the form:

$$C_{\alpha\beta\gamma} \mid \delta = C_{hij} \mid {}_k B^h_\alpha B^i_\beta B^j_\gamma B^k_\delta + F\rho^2(h_{\beta\gamma}h_{\alpha\delta} + h_{\alpha\gamma}h_{\beta\delta} + h_{\alpha\beta}h_{\gamma\delta}). \qquad \text{...(5.5)}$$

Substituting this expression in

$$T_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma} \mid \delta + C_{\beta\gamma\delta}l_\alpha + C_{\alpha\gamma\delta}l_\beta + C_{\alpha\beta\delta}l_\gamma + C_{\alpha\beta\gamma}l_\delta$$

and simplifying we get

$$T_{\alpha\beta\gamma\delta} = T_{hijk} B^h_\alpha B^i_\beta B^j_\gamma B^k_\delta + F\rho^2 (h_{\beta\gamma}h_{\alpha\delta} + h_{\alpha\gamma}h_{\beta\delta} + h_{\alpha\beta}h_{\gamma\delta}). \qquad \text{...(5.6)}$$

The space F_n is said to satisfy T-condition if and only if $T_{hijk} = 0$.

Theorem 5.1 — If the C-reducible Finsler space F_n satisfies T-condition then the necessary and sufficient condition that its hypersurface F_{n-1} will satisfy T-condition is that the vector field C_i is tangential to the space F_{n-1}.

PROOF : Using the conditions $T_{hijk} = 0$, $T_{\alpha\beta\gamma\delta} = 0$ in eqn. (5.6) we find that either

(a) $\rho = C_i N^i = 0$ or (b) $h_{\beta\gamma}h_{\alpha\delta} + h_{\alpha\gamma}h_{\beta\delta} + h_{\alpha\beta}h_{\gamma\delta} = 0$.

The relation (b), after transvection with respect to $g^{\beta\gamma}$ will give $h_{\alpha\delta} = 0$. Therefore, the eqn. (2.3) yields $C_{\alpha\beta\gamma} = 0$. This is impossible as the space F_{n-1} is non-Riemannian. This proves the theorem.

The Theorems 3.1, 3.2 and 5.1 yield the following:

Theorem 5.2 — If a C-reducible Landsberg space F_n and its hypersurface F_{n-1} satisfy T-conditions then the hypersurface is Landsberg and its induced and intrinsic connection parameters are equal and independent of direction element $\dot{u}$.

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