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A NOTE ON THE SEMI-CONTINUITY PROPERTIES OF THE FARTHEST POINT MAP

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In this note, a necessary and sufficient condition for the farthest point map to be upper semi-continuous is given. In addition, another sufficient condition is obtained. The corresponding results for lower semi-continuity are provided. These results are analogues of the known results for the nearest point map.

1. INTRODUCTION

Blatter (1969) and Narang (1978) have studied the semi-continuity properties of the farthest point map in normed linear spaces and metric spaces, respectively. Here, a necessary and sufficient condition for the farthest point map in a normed linear space to be upper semi-continuous (u.s.c.) is proved. A sufficient condition for the farthest point map to be u.s.c. is also established. Corresponding results for the lower semi-continuity (l.s.c.) of the farthest point map are given. These results turn out to be exact analogues of the known results for the nearest point map in normed linear spaces.

Let E be a normed linear space and let G be a subspace of E. Let $x \in E$. Then the point $g_0 \in G$ is said to be 'a farthest point' of x if

$$\| x - g_0 \| = \sup \{ \| x - g \| : g \in G\}.$$

The set of all farthest points of x in G is denoted by $\phi_G(x)$. If $\phi_G(x) \neq \phi$, for all $x \in E$, then G is called 'a remotal subspace' of E. If G is a remotal subspace, then the set-valued map $\phi_G : x \to \phi_G(x)$ is called the farthest point map.

Let X and Y be metric spaces. A set valued map $f: X \to 2^Y$ is called u.s.c. (respectively l.s.c) if and only if the set

$$\{x \in X \mid f(x) \cap N \neq \emptyset\}$$

is closed for each closed subset N of Y (respectively, open for each open subset N of Y).

From the definition of the upper semi-continuity of a set-valued function, it is clear that ϕ_G is u.s.c if and only if for each closed subset N of G, the subset

$$\bigcup_{y \in N} \{x \in E \mid y \in \phi_G(x)\}$$

is closed.

2. Upper Semi-continuity of the Farthest Point Map

Godini (1977) has proved a necessary and sufficient condition for the nearest point map to be u.s.c. The following theorem for the farthest point map is an analogue of his result.

Theorem 1 — If G is a remotal subspace of the normed linear space E, then ϕ_G is u.s.c. if and only if for each closed subset N of G, $N + \phi_G^{-1}(0)$ is closed.

Proof : First it is claimed that $y \in \phi_G(x)$ implies $x - y \in \phi_G^{-1}(0)$. Indeed, $y \in \phi_G(x)$ implies

$$\| x - y \| \geqslant \| x - g \|, \text{ for all } g \in G$$

i.e. $$\| x - y \| \geqslant \| x - y - g \|, \text{ for all } g \in G.$$

i.e. $$0 \in \phi_G(x - y). \quad \text{Hence } x - y \in \phi_G^{-1}(0).$$

Let $z \in \phi_G^{-1}(0)$ and $y \in G$. $z \in \phi_G^{-1}(0)$ implies $\| z \| \geqslant \| z - g \|$, for all $g \in G$.

i.e. $$\| y + z - y \| \geqslant \| y + z - g \|, \text{ for all } g \in G.$$

i.e. $$y \in \phi_G(y + z).$$

Let $x \in E$ and $y \in G$ such that $y \in \phi_G(x)$. Then $x \in y + \phi_G^{-1}(0)$.

i.e. $$\{x \in E \mid y \in \phi_G(x)\} \subset y + \phi_G^{-1}(0).$$

Let $z \in \phi_G^{-1}(0)$. Since $y \in G$, $y \in \phi_G(y + z)$.

i.e. $$y + \phi_G^{-1}(0) \subset \{x \in E \mid y \in \phi_G(x)\}.$$

Therefore

$$N + \phi_G^{-1}(0) = \bigcup_{y \in N} \{x \in E \mid y \in \phi_G(x)\}.$$

Hence it follows that ϕ_G is u.s.c. if and only if $N + \phi_G^{-1}(0)$ is closed for each closed subset N of G. This completes the proof.

If G is remotal, then it is easy to verify that

$$\phi_G(x + g) = \phi_G(x) + g$$

for all $x \in E$ and for all $g \in G$.

Theorem 2 — Let G be a closed remotal subspace of the normed linear space E such that $E = G \oplus L$ (topological direct sum). Then ϕ_G is u.s.c. if for each compact subset $A \subset L$, the subset $\phi_G(A)$ is a compact subset of G.

PROOF: Let N be a closed subset of G and $x_n \in \{x \in E \mid \phi_G(x) \cap N \neq \emptyset\}$ such that $x_n \to x_0$. Since $E = G \oplus L$, let $x_n = y_n + z_n (n = 0, 1, 2, \ldots)$, where $y_n \in G$ and $z_n \in L$. Let P_1 be the projection from E onto G such that $P_1(x_n) = y_n$. Then, from the continuity of P_1, it follows that $y_n \to y_0$ and $z_n \to z_0$. As

$$\phi_G(x_n) \cap N \neq \emptyset,$$

let $\bar{g}_n \in \phi_G(x_n) \cap N (n = 1, 2, \ldots)$. Since $\phi_G(x_n) = y_n + \phi_G(z_n)$, let $g_n \in \phi_G(x_n)$ such that $\bar{g}_n = y_n + g_n$. The subset $A = \{z_n\}_{n=0}^{\infty}$ being compact in L, by hypothesis, $\phi_G(A)$ is compact. Since $g_n \in \phi_G(A)$, there exists a convergent subsequence $\{g_{n_i}\}$ of $\{g_n\}$ such that $g_{n_i} \to g_0 \in G$. Now

$$g_{n_i} \in \phi_G(z_{n_i}) \Rightarrow z_{n_i} - g_{n_i} \in \phi_G^{-1}(0).$$

Therefore

$$\lim_i (z_{n_i} - g_{n_i}) = z_0 - g_0 \in \phi_G^{-1}(0).$$

i.e. $g_0 \in \phi_G(z_0)$. Since $\phi_G(x_0) = y_0 + \phi_G(z_0)$, $y_0 + g_0 \in \phi_G(x_0)$. From the facts that $\bar{g}_{n_i} = y_{n_i} + g_{n_i} \to y_0 + g_0$ and N is closed, it follows that $y_0 + g_0 \in N$. i.e. $y_0 + g_0 \in \phi_G(x_0) \cap N$. Therefore

$$x_0 \in \{x \in E \mid \phi_G(x) \cap N \neq \emptyset\}.$$

Hence

$$\{x \in E \mid \phi_G(x) \cap N \neq \emptyset\}$$

is closed for each closed subset N of G.

3. Lower Semi-Continuity of the Farthest Point Map

Similar to Theorem 1, one can prove the following

Theorem 3 — If G is a remotal subspace of the normed linear space E, then ϕ_G is l.s.c. if and only if for each open set D of G, $D + \phi_G^{-1}(0)$ is open in E.

Theorem 4 — Let G be a closed remotal subspace of the normed linear space E such that $E = G \oplus L$ (topological direct sum). Then ϕ_G is l.s.c. if for each relatively compact subset of A of L, $\overline{\phi_G(A)} = \phi_G(\bar{A})$.

PROOF : If ϕ_G is not l.s.c. then there exist an open subset $N \subset G$ and $x_n \to x_0$ such that $\phi_G(x_n) \cap N = \emptyset (n = 1, 2, ...)$ and $\phi_G(x_0) \cap N \neq \emptyset$. Since N is open, let $\bar{g}_0 \in \phi_G(x_0) \cap N$ and $\epsilon > 0$ such that

$$\{g \in G \mid \| g - \bar{g}_0 \| < \epsilon\} \subset N.$$

From $E = G \oplus L$, there exist $y_n \in G$ and $z_n \in L$ such that $x_n = y_n + z_n$ $(n = 0, 1, 2, ...)$. Since $x_n \to x_0$, $y_n \to y_0$ and $z_n \to z_0$. From

$$\phi_G(x_0) = y_0 + \phi_G(z_0)$$

it follows that there exists an element $g_0 \in \phi_G(z_0)$ such that

$$y_0 + g_0 = \bar{g}_0.$$

If $g_0 \in \phi_G(z_{n_i})$, $(i = 1, 2, ...)$, then $y_{n_i} + g_0 \in \phi_G(x_{n_i})$. Since $y_{n_i} \to y_0$, for sufficiently large i, $y_{n_i} + g_0 = \epsilon_0 N \cap \phi_G(x_{n_i})$, which is a contradiction. So it can be supposed that $g_0 \in \phi_G(z_n)$ $(n = 1, 2, ...)$. Since $A = \{z_n\}_{n=1}^{\infty}$ is relatively compact in L, by hypothesis, $\phi_G(\bar{A}) = \overline{\phi_G(A)}$. From $g_0 \in \phi_G(z_0) \subset \phi_G(\bar{A})$, it follows that there exist $g_{n_i} \in \phi_G(z_{n_i})$ such that $g_{n_i} \to g_0$ $(i = 1, 2, ...)$. Now from the facts that

$$y_{n_i} + g_{n_i} \in \phi_G(x_{n_i})$$

and

$$\| y_{n_i} + g_{n_i} - \bar{g}_0 \| \leqslant \| y_{n_i} - y_0 \| + \| g_{n_i} - g_0 \| < \epsilon,$$

it follows that

$$y_{n_i} + g_{n_i} \in \phi_G(x_{n_i}) \cap N,$$

which is a contradiction. This completes the proof.

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