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INNER FUNCTIONS AND COMPOSITION OPERATORS ON A HARDY SPACE

R. K. SINGH

Department of Mathematics, University of Jammu, Canal Road, Jammu 180001

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If L is the linear fractional transformation defined by $L(z) = i(1 + z)/(1 - z)$ and T is an analytic mapping from the unit disc D into itself, then the function $T =$ Loto L^{-1} maps the upper half plane π^+ into itself. A necessary condition for T to induce a composition operator on $H^2(\pi^+)$ is reported in this paper.

1. INTRODUCTION

Let X be a non-empty set and $H(X)$ be a functional Hilbert space on X. Then every mapping T from X into itself induces a linear transformation C_T on $H(X)$ into the space of all complex valued functions on X defined by $C_T f = f \circ T$ for every f in $H(X)$. If the range of C_T is a linear manifold of $H(X)$, then by an application of the closed graph theorem, C_T is a continuous operator on $H(X)$ which we call a composition operator induced by T. Nordgren (1968, 1978), Schwartz (1969) and Singh (1975) have studied these operators on different functional Hilbert spaces.

In this note we are interested in the case when $X = \pi^+$, the upper half-plane, and $H = H^2(\pi^+)$, where $H^2(\pi^+) = \{f \mid f$ is an analytic function on π^+ and $\sup_{y>0} \int_{-\infty}^{\infty} | f(x + iy) |^2 \, dx < \infty\}$. If L is the linear fractional transformation defined by $L(z) = i(1 + z)/(1 - z)$, then it maps the unit disc D onto π^+ and L^{-1}, the inverse of L takes π^+ onto D (Hoffman 1962). If t is an analytic function from D into itself, then it is shown in Schwartz (1969) that C_t is a bounded operator on $H^2(D)$. Every such t gives rise to an analytic function T from π^+ into itself defined by $T(w) =$ (Loto L^{-1}) (w) for every w in π^+. We have proved (see Singh 1975) that C_T is a bounded operator on $H^2(\pi^+)$ if and only if $M_\beta C_t$ is a bounded operator on $H^2(D)$, where M_β is the multiplication operator induced by the function β which is defined as $\beta(z) = (1 - t(z))/(1 - z)$. In this paper t is taken to be an inner function (a holomorphic function on D of the unit modulus almost everywhere on the unit circle), and a necessary condition for C_T to be a continuous operator on $H^2(\pi^+)$ is reported.

2. INNER FUNCTIONS AND COMPOSITION OPERATORS

To arrive at the main result of this note we shall need the following lemmas.

Lemma 1 — If t is an inner function, then ran C_t (the range of C_t) is a closed subspace of $H^2(D)$.

PROOF : Let $\{C_t f_n\}$ be a sequence in ran C_t converging to a function h. Since t is inner, by a theorem of Nordgren (1969) we get

$$\| C_t(f_m - f_n) \| \geqslant a \| f_m - f_n \|$$

for some $a > 0$. This shows that $\{f_n\}$ is a Cauchy sequence. The completeness of $H^2(D)$ gives a function g in $H^2(D)$ such that the sequence $\{f_n\}$ converges to g in norm. Since C_t is continuous, the sequence $\{C_t f_n\}$ converges to $C_t g$ in norm. Thus we can conclude that $h = C_t g$, which shows that h belongs to ran C_t. Thus ran C_t is a closed subset of $H^2(D)$. Since the range of every operator is a subspace (a linear manifold), the proof of the lemma is completed.

Now, $H^2(D)$ is a functional Hilbert space and ran C_t is closed in $H^2(D)$, and hence ran C_t is also a functional Hilbert space. Let K and K' be the reproducing kernels of $H^2(D)$ and ran C_t respectively. Then by Problem 30 of Halmos (1967)

$$K(x, y) = 1/(1 - xy).$$

Lemma 2 — Let t be an inner function and let $z \in D$. Then

$$\| K_{t(z)} \| \leqslant \| C_t \| \| K'_z \|.$$

PROOF : If $z \in D$, then by the definition of K and K'

$$(f \circ t)(z) = \langle f \circ t, K'_z \rangle$$

$$= \langle f, K_{t(z)} \rangle$$

for every f in $H^2(D)$. Hence

$$| \langle f, K_{t(z)} \rangle | \leqslant \| C_t f \| \| K'_z \|$$

$$\leqslant \| C_t \| \| K'_z \| \| f \|.$$

From this we have

$$\sup \{ | \langle f, K_{t(z)} \rangle | / \| f \| : \| f \| \neq 0\} \leqslant \| C_t \| \| K'_z \|,$$

and hence

$$\| K_{t(z)} \| \leqslant \| C_t \| \| K'_z \|.$$

This completes the proof of the lemma.

If t is an inner function, then $\lim_{r \to 1} t(re^{i\theta})$ exists almost everywhere. We shall denote it by $t_*(e^{i\theta})$. The function t_* is known as the non-tangential limit of t. The principal result of this paper is given in the following theorem.

Theorem — If t is an inner function and $T = \text{Loto } L^{-1}$, then C_T is a bounded operator on $H^2(\pi^+)$ only if $t_*(1) = 1$.

PROOF: Let C_T be a bounded operator on $H^2(D)$. Then by a theorem of Singh (1975) $M_\beta C_t$ is a bounded operator on $H^2(D)$. Hence

$$\| M_\beta C_t f \| \leqslant \| M_\beta C_t \| \, \| f \|$$

for every f in $H^2(D)$. Since C_t is bounded away from zero, there exists $a > 0$ such that

$$\| f \| \leqslant a \| C_t f \|$$

for every f in $H^2(D)$. Hence

$$\| M_\beta C_t f \| \leqslant a \| M_\beta C_t \| \, \| C_t f \|.$$

This shows that M_β is a bounded linear transformation from ran C_t into $H^2(D)$. Now, we have

$$\begin{aligned} | \beta(z) K_z'(z) | &= | (\beta \,.\, K_z')(z) | \\ &= | \langle M_\beta K_z', K_z \rangle | \\ &\leqslant \| M_\beta \| \, \| K_z' \| \, \| K_z \|. \end{aligned}$$

An application of lemma 2 yields

$$| \beta(z) | \leqslant \| M_\beta \| \, \| C_t \| \, (\| K_z \| / \| K_{t(z)} \|).$$

By Problem 30 of Halmos (1967) we have

$$\| K_z \|^2 = 1/(1 - | z |^2).$$

Hence, letting $N = \| M_\beta \| \, \| C_t \|$ in the above inequality we get

$$| \beta(z) | \leqslant N((1 - | t(z) |^2)/(1 - | z |^2))^{1/2}.$$

From this we get

$$| 1 - t(z) | \leqslant N((| 1 - z |)/(1 - | z |^2))^{1/2} \,.\, (| 1 - z |)^{1/2}.$$

Since $| 1 - z | /(1 - | z |^2)$ is bounded in a kite shaped region with vertex at 1 (Duren 1970), the right-hand side of the above inequality tends to zero as z tends to 1 non-tangentially. This shows that $t_*(1) = 1$. This completes the proof of the theorem.

In the argument above, we considered an inner function t in order to be sure of the continuity of M_β on ran C_t. If t is not inner, then C_t is not necessarily bounded away from zero, and hence we can not conclude that M_β is continuous on ran C_t. In this regard we cite the following example.

Example : Lt $t(z) = (\frac{1}{2})z$. Then t is not inner, and C_t is continuous on $H^2(D)$. Let $e_n(z) = z^n$ be the nth basis vector. Then

$$\| C_t e_n \| = \| e_n \circ t \|$$
$$= (1/2^n) \| e_n \|.$$

This shows that C_t is not bounded away from zero.

Note : If t is any analytic function from D into itself, then the above theorem is true as we have been able to prove this using a technique different from the one given in this paper. We have a guess that the theorem is true other way also in case of an inner function. This is the main reason for developing the technique of the proof taking only inner functions.

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