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ON GENERATING RELATIONS OF GENERAL CHARACTER

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This paper deals with the generating relations involving the basic and ordinary hypergeometric functions of several variables. These results arise as special cases of the general formulae discussed in the paper.

1. INTRODUCTION

Recently, Exton (1976, Chap. 6) has given two theorems on the unification of several generating relations of different types. Following Exton (1976), we derive certain generating relations of general nature which as special cases give the generating relations involving

$$\Phi_A^{[r]},\ \Phi_B^{[r]},\ F_A,\ F_B,\ \ldots,\ \text{etc.}$$

defined below.

Notation and Definitions

$$\left.\begin{aligned}(a)_n &= a(a+1)\ldots(a+n-1),\ n \geqslant 1\\ (a)_0 &= 1.\end{aligned}\right\} \qquad \ldots(1.1)$$

$$\left.\begin{aligned}[a]_n &= (1-a)(1-aq)\ldots(1-aq^{n-1})\\ [a]_0 &= 1;\ |\,q\,| < 1\end{aligned}\right\} \qquad \ldots(1.2)$$

in this case, a is written for q^a.

$$\Phi_A^{[r]}\,[a : b_1, \ldots, b_r; c_1, \ldots, c_r; x_1, \ldots, x_r]$$

$$= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{[a]_{m_1+\cdots+m_r}\,[b_1]_{m_1}\ldots[b_r]_{m_r}}{[c_1]_{m_1}\ldots[c_r]_{m_r}}\ \frac{x_1^{m_1}\ldots x_r^{m_r}}{[1]_{m_1}\ldots[1]_{m_r}} \qquad \ldots(1.3)$$

$$\Phi_B^{*[r]}\,[a_1, \ldots, a_r, b_1, \ldots, b_r; c : x_1, \ldots, x_r] =$$

(*equation continued on p.* 1302)

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$$= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{[a_1]_{m_1} \ldots [a_r]_{m_r} [b_1]_{m_1} \ldots [b_r]_{m_r}}{[c]_{m_1+\cdots+m_r}}$$

$$\times \frac{x_1^{m_1} \ldots x_r^{m_r}}{[1]_{m_1} \ldots [1]_{m_r}} q^{(m_1+\cdots+m_r)(m_1+\cdots+m_r-1)/2} \quad \ldots(1.4)$$

$$\Phi_C^{[r]} [a, b : c_1, \ldots, c_r; x_1, \ldots, x_r]$$

$$= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{[a]_{m_1+\cdots+m_r} [b]_{m_1+\cdots+m_r}}{[c_1]_{m_1} \ldots [c_r]_{m_r}} \frac{x_1^{m_1} \ldots x_r^{m_r}}{[1]_{m_1} \ldots [1]_{m_r}} \quad \ldots(1.5)$$

$$\Phi_D^{[r]} [a : b_1, \ldots, b_r; c : x_1, \ldots, x_r]$$

$$= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{[a]_{m_1+\cdots+m_r} [b_1]_{m_1} \ldots [b_r]_{m_r}}{[c]_{m_1+\cdots+m_r}} \frac{x_1^{m_1} \ldots x_r^{m_r}}{[1]_{m_1} \ldots [1]_{m_r}} \quad \ldots(1.6)$$

$$\Phi_{[2]}^{*[r]} [a_1, \ldots, a_r; b : x_1, \ldots, x_r]$$

$$= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{[a_1]_{m_1} \ldots [a_r]_{m_r}}{[b]_{m_1+\cdots+m_r}} \frac{x_1^{m_1} \ldots x_r^{m_r}}{[1]_{m_1} \ldots [1]_{m_r}} q^{(m_1+\cdots+m_r)(m_1+\cdots+m_r-1)/2} \quad \ldots(1.7)$$

$$\Psi_{[2]}^{[n]} [a : b_1, \ldots, b_r; x_1, \ldots, x_r]$$

$$= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{[a]_{m_1+\cdots+m_r}}{[b_1]_{m_1} \ldots [b_r]_{m_r}} \frac{x_1^{m_1} \ldots x_r^{m_r}}{[1]_{m_1} \ldots [1]_{m_r}} \quad \ldots(1.8)$$

and

$$e_q^* (x) = \sum_{n=0}^{\infty} \frac{x^n}{[1]_n} q^{n(n-1)/2}. \quad \ldots(1.9)$$

As $q \to 1$, (1.3), (1.4), ... will define the corresponding functions F_A, F_B, ..., etc.

We shall also need the following formulae (Slater 1966):

$$\frac{q^{nr} [q^{-n}]_r}{[1]_n} = \frac{(-)^r q^{r(r-1)/2}}{[1]_{n-r}} \quad \ldots(1.10)$$

and

$$[q^{1-N}/a]_n = \frac{[a]_N \, q^{n(n+1)/2}}{(-a)^n \, q^{Nn} \, [a]_{N-n}}. \qquad \ldots(1.11)$$

2. Generating Relations

Theorem 1 — For arbitrary complex coefficients $C[k_1, \ldots, k_r]$, $K_j \geqslant 0$, $1 \leqslant j \leqslant r$, let

$$S^{[r]}[\lambda : a_1, \ldots, a_r; \mu : b_1, \ldots, b_r; z_1, \ldots, z_r]$$

$$= \sum_{k_1, \ldots, k_r = 0}^{\infty} \frac{[\lambda]_{k_1 + \cdots + k_r} [a_1]_{k_1} \ldots [a_r]_{k_r}}{[\mu]_{k_1 + \cdots + k_r} [b_1]_{k_1} \ldots [b_r]_{k_r}} C[k_1, \ldots, k_r] \frac{z_1^{k_1} \ldots z_r^{k_r}}{[1]_{k_1} \ldots [1]_{k_r}}. \qquad \ldots(2.1)$$

Then

$$\sum_{n_1, \ldots, n_r = 0}^{\infty} \frac{t_1^{n_1} \ldots t_r^{n_r}}{[1]_{n_1} \ldots [1]_{n_r}} q^{(n_1^2 + \ldots + n_r^2)/2}$$

$$\times S^{[r]}[- : -n_1, \ldots, -n_r; - : -, \ldots, -; z_1, \ldots, z_r]$$

$$= \{e_q^*(q^{1/2}t_1) \ldots e_q^*(q^{1/2}t_r)\} S^{[r]}[- : -, \ldots, -; - : -, \ldots, -;$$

$$-q^{-1/2}z_1t_1, \ldots, -q^{-1/2}z_rt_r], \qquad \ldots(2.2)$$

$$\sum_{n_1, \ldots, n_r = 0}^{\infty} \frac{t_1^{n_1} \ldots t_r^{n_r}}{[1]_{n_1 + \cdots + n_r}} q^{(n_1 + \cdots + n_r)^2/2}$$

$$\times S^{[r]}[-n_1 - \ldots - n_r : -, \ldots, -; - : -, \ldots, -; z_1, \ldots, z_r]$$

$$= \Phi_{[2]}^{*[r]}[1, \ldots, 1; 1 : q^{1/2}t_1, \ldots, q^{1/2}t_r]$$

$$\times S^{[r]}[- : -, \ldots, -; - : -, \ldots, -; -q^{-1/2}z_1t_1, \ldots, -q^{-1/2}z_rt_r] \qquad \ldots(2.3)$$

and

$$\sum_{n_1, \ldots, n_r = 0}^{\infty} \frac{[\lambda]_{n_1 + \cdots + n_r}}{[1]_{n_1 + \cdots + n_r}} (t_1^{n_1} \ldots t_r^{n_r})$$

$$\times S^{[r]}[-n_1 - \ldots - n_r : -, \ldots, -; 1 - \lambda - n_1 - \ldots - n_r :$$

$$-, \ldots, -; z_1, \ldots, z_r]$$

$$= \Phi_D^{[r]}[\lambda : 1, \ldots, 1; 1 : t_1, \ldots, t_r] S^{[r]}[- : -, \ldots, -; - :$$

$$-, \ldots, -; q^{\lambda-1}z_1t_1, \ldots, q^{\lambda-1}z_rt_r]. \qquad \ldots(2.4)$$

Proof of (2.2)

In (2.2) putting the value of $S^{(r)}$ from (2.1), the left-hand side will become

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \{[-n_1]_{k_1} \ldots [-n_r]_{k_r}\}$$

$$\times\, C\,[k_1, \ldots, k_r]\, q^{(n_1^2+\cdots+n_r^2)/2} \frac{t_1^{n_1} \ldots t_r^{n_r}}{[1]_{n_1} \ldots [1]_{n_r}} \frac{z_1^{k_1} \ldots z_r^{k_r}}{[1]_{k_1} \ldots [1]_{k_r}}.$$

Now the application of (1.10), will yield

$$= \sum_{\substack{n_1,\ldots,n_r,\\ k_1,\ldots,k_r=0}}^{\infty} C[k_1, \ldots, k_r]\, q^{n_1(n_1-1)/2+\cdots+n_r(n_r-1)/2}$$

$$\times \frac{(q^{1/2}t_1)^{n_1} \ldots (q^{1/2}t_r)^{n_r}}{[1]_{n_1} \ldots [1]_{n_r}} \frac{(-q^{-1/2}z_1t_1)^{k_1} \ldots (-q^{-1/2}z_rt_r)^{k_r}}{[1]_{k_1} \ldots [1]_{k_r}}$$

which after some simplification gives (2.2).

For the sake of brevity and compactness of the paper the proofs of the other results are not given because they can be proved in a similar way.

Theorem 2 — If

$$S^{(r)}(\lambda : a_1, \ldots, a_r; \mu : b_1, \ldots, b_r; z_1, \ldots, z_r)$$

$$= \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(\lambda)_{k_1+\cdots+k_r} (a_1)_{k_1} \ldots (a_r)_{k_r}}{(\mu)_{k_1+\cdots+k_r} (b_1)_{k_1} \ldots (b_r)_{k_r}} C(k_1, \ldots, k_r) \frac{x_1^{k_1} \ldots x_r^{k_r}}{k_1! \ldots k_r!} \qquad \ldots(2.5)$$

where $C(k_1, \ldots, k_r)$ are arbitrary complex coefficients such that $k_j \geqslant 0$, $1 \leqslant j \leqslant r$.

Then

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \frac{n_1! \ldots n_r!}{(\lambda)_{n_1+\cdots+n_r}} (t_1^{n_1} \ldots t_r^{n_r})$$

$$\times\, S^{(r)}(1-\lambda-n_1-\ldots-n_r : -, \ldots, -; - : -n_1, \ldots, -n_r; z_1, \ldots, z_r)$$

$$= F_B(1, \ldots 1; 1, \ldots, 1; \lambda : t_1, \ldots, t_r)$$

$$\times\, S^{(r)}(- : -, \ldots, -; - : -, \ldots, -; z_1t_1, \ldots, z_rt_r) \qquad \ldots(2.6)$$

and

$$\sum_{n_1,\ldots,n_r=0}^{\infty} (n_1! \ldots n_r!)\,(t_1^{n_1} \ldots t_r^{n_r})$$

$$\times\, S^{(r)}(- : -, \ldots, -; - : -n_1, \ldots, -n_r; z_1, \ldots, z_r)$$

$$= \{{}_2F_0(1, 1; -; t_1) \ldots {}_2F_0(1, 1; -; t_r)\}$$

$$\times\, S^{(r)}(- : -, \ldots, -; - : -, \ldots, -; -z_1t_1, \ldots, -z_rt_r). \quad \ldots(2.7)$$

The expression (2.7) is a formal result since the series representation of the ${}_2F_0$ is divergent.

Proof of (2.6)

Substituting the value of $S^{(r)}$ from (2.5) in the left-hand side of (2.6), we get

$$\text{L.H.S.} \quad = \sum_{n_1,\ldots,n_r=0}^{\infty} \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \frac{(1-\lambda-n_1-\ldots-n_r)_{k_1+\ldots+k_r}}{(\lambda)_{n_1+\ldots+n_r}}$$

$$\times \frac{n_1! \ldots n_r!}{(-n_1)_{k_1} \ldots (-n_r)_{k_r}} (t_1^{n_1} \ldots t_r^{n_r}) \left(\frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!}\right) C(k_1, \ldots, k_r)$$

now applying the results (1.10) and (1.11) (for $q \to 1$), we obtain

$$= \sum_{n_1,\ldots,n_r=0}^{\infty} \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \frac{(n_1-k_1)! \ldots (n_r-k_r)!}{(\lambda)_{(n_1+\ldots+n_r)-(k_1+\ldots+k_r)}}$$

$$\times\, C(k_1, \ldots, k_r)\,(t_1^{n_1} \ldots t_r^{n_r}) \left(\frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!}\right)$$

which gives (2.6).

In a similar way (2.7) can also be proved.

3. Particular Cases

By giving different values to $C[k_1, \ldots, k_r]$ and $C(k_1, \ldots, k_r)$ in the above formulae a number of generating relations can be obtained. Here we shall quote only a few interesting examples.

(i) The substitution $C[k_1, \ldots, k_r] = \dfrac{[\mu]_{k_1+\ldots+k_r}}{[a_1]_{k_1} \ldots [a_r]_{k_r}}$ in (2.2) and (2.3), will give

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \frac{t_1^{n_1} \ldots t_r^{n_r}}{[1]_{n_1} \ldots [1]_{n_r}} q^{(n_1^2+\ldots+n_r^2)/2}$$

$$\times \Phi_A^{[r]} [\mu : -n_1, \ldots, -n_r; a_1, \ldots, a_r; z_1, \ldots, z_r]$$

$$= \{e_q^* (q^{1/2}t_1) \ldots e_q^* (q^{1/2}t_r)\} \Psi_{[2]}^{[r]} [\mu : a_1, \ldots, a_r; -q^{-1/2}z_1t_1, \ldots, -q^{-1/2}z_rt_r] \quad ..(3.1)$$

and

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \frac{t_1^{n_1} \ldots t_r^{n_r}}{[1]_{n_1+\ldots+n_r}} q^{(n_1+\cdots+n_r)^2/2}$$

$$\times \Phi_C^{[r]} [-n_1 - \ldots -n_r, \mu : a_1, \ldots, a_r; z_1, \ldots, z_r]$$

$$= \Phi_{[2]}^{*[r]} [1, \ldots, 1; 1 : q^{1/2}t_1, \ldots, q^{1/2}t_r]$$

$$\times \Psi_{[2]}^{[r]} [\mu : a_1, \ldots, a_r; -q^{-1/2}z_1t_1, \ldots, -q^{-1/2}z_rt_r] \quad ...(3.2)$$

respectively.

(ii) Now by taking $C[k_1, \ldots, k_r] = \frac{[a_1]_{k_1} \ldots [a_r]_{k_r}}{[\mu]_{k_1+\cdots+k_r}}$ in (2.2) and (2.3),

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \frac{t_1^{n_1} \ldots t_r^{n_r}}{[1]_{n_1} \ldots [1]_{n_r}} q^{(n_1^2+\ldots+n_r^2)/2}$$

$$\times \Phi_B^{[r]} [-n_1, \ldots, -n_r; a_1, \ldots, a_r; \mu : z_1, \ldots, z_r]$$

$$= \{e_q^* (q^{1/2}t_1) \ldots e_q^* (q^{1/2}t_r)\} \Phi_{[2]}^{[r]} [a_1, \ldots, a_r; \mu : -q^{-1/2}z_1t_1, \ldots, -q^{-1/2}z_rt_r] \quad ...(3.3)$$

and

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \frac{t_1^{n_1} \ldots t_r^{n_r}}{[1]_{n_1+\ldots+n_r}} q^{(n_1+\cdots+n_r)^2/2}$$

$$\times \Phi_D^{[r]} [-n_1 - \ldots -n_r : a_1, \ldots, a_r; \mu : z_1, \ldots, z_r]$$

$$= \Phi_{[2]}^{*[r]} [1, \ldots, 1; 1 : q^{1/2}t_1, \ldots, q^{1/2}t_r] \Phi_{[2]}^{[r]} [a_1, \ldots, a_r; \mu : -q^{-1/2}z_1t_1, \ldots, -q^{-1/2}z_rt_r] \quad ...(3.4)$$

respectively.

(iii) Again setting $C[k_1, ..., k_r] = [a_1]_{k_1} ... [a_r]_{k_r}$ in (2.4), it follows that

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \frac{[\lambda]_{n_1+\ldots+n_r}}{[1]_{n_1+\ldots+n_r}} (t_1^{n_1} \ldots t_r^{n_r}) \times \Phi_D^{[r]} [-n_1 - \ldots - n_r : a_1, \ldots, a_r; 1 - \lambda - n_1 - \ldots - n_r : z_1, \ldots, z_r]$$

$$= \{{}_1\Phi_0 [a_1; -; q^{\lambda-1}z_1t_1] \ldots {}_1\Phi_0 [a_r; -; q^{\lambda-1}z_rt_r]\} \times \Phi_D^{[r]} [\lambda : 1, \ldots, 1; 1 : t_1, \ldots, t_r]. \quad \ldots(3.5)$$

(iv) If in (2.6) $C(k_1, ..., k_r) = (a_1)_{k_1} ... (a_r)_{k_r}$, then we can write

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \frac{n_1! \ldots n_r!}{(\lambda)_{n_1+\ldots+n_r}} (t_1^{n_1} \ldots t_r^{n_r}) \times F_A(1 - \lambda - n_1 - \ldots - n_r; a_1, \ldots, a_r; -n_1, \ldots, -n_r; z_1, \ldots, z_r)$$

$$= \{(1 - z_1t_1)^{-a_1} \ldots (1 - z_rt_r)^{-a_r}\} \times F_B(1, \ldots, 1; 1, \ldots, 1; \lambda : t_1, \ldots, t_r). \quad \ldots(3.6)$$

(v) Lastly, $C(k_1, ..., k_r) = (\mu)_{k_1+\ldots+k_r}$ in (2.7), will give the formal result

$$\sum_{n_1,\ldots,n_r=0}^{\infty} (n_1! \ldots n_r!) (t_1^{n_1} \ldots t_r^{n_r}) \Psi_2^{(r)} (\mu : -n_1, \ldots, -n_r; z_1, \ldots, z_r)$$

$$= \{{}_2F_0(1, 1; -; t_1) \ldots {}_2F_0(1, 1; -; t_r)\} \times F_A(\mu : 1, \ldots, 1; 1, \ldots 1; -z_1t_1, \ldots, -z_rt_r). \quad \ldots(3.7)$$

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REFERENCES

Exton, H. (1976). Multiple Hypergeometric Functions and Applications. Ellis Horwood, Chichester.

Slater, L. J. (1966). Generalized Hypergeometric Functions. Cambridge University Press, London.