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ON CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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Let $S_\lambda(\alpha, \beta)$ denote the class of functions $f(z) = z + a_2z^2 + \ldots$ which are regular in the unit disc $E = \{z : |z| < 1\}$ and satisfy the condition

$$\left|\frac{(f(z)/g(z)) - 1}{(\lambda f(z)/g(z)) + 1}\right| < \beta, \quad 0 \leqslant \lambda \leqslant 1, 0 < \beta \leqslant 1, z \in E$$

where $g(z)$ is starlike of order $\alpha(0 \leqslant \alpha < 1)$ in E. By $K_\lambda(\alpha, \beta)$ we denote the class of functions $F(z) = \frac{1}{z} + c_0 + c_1z + \ldots$ regular in $0 < |z| < 1$ and satisfying the condition

$$\left|\frac{(F(z)/G(z)) - 1}{(\lambda(F(z)/G(z)) + 1}\right| < \beta, \quad 0 \leqslant \lambda \leqslant 1, 0 < \beta \leqslant 1$$

where $G(z)$ is regular and starlike of order $\alpha(0 \leqslant \alpha < 1)$ in $0 < |z| < 1$. In this paper we obtain distortion theorem and coefficient estimates of the class $S_\lambda(\alpha, \beta)$. The radius of close to convexity for the class $S_0(0, \beta)$ is obtained. Further we obtain the radii of starlikeness for the classes $S_\lambda(\alpha, \beta)$ and $K_\lambda(\alpha, \beta)$.

1. INTRODUCTION

Let $S_\lambda(\alpha, \beta)$ be the class of functions

$$f(z) = z + a_2z^2 + \ldots \quad \ldots(1)$$

which are regular in the unit disc $E = \{z : |z| < 1\}$ and satisfy the condition

$$\left|\left(\frac{f(z)}{g(z)} - 1\right)\Big/\left(\lambda\frac{f(z)}{g(z)} + 1\right)\right| < \beta, \quad z \in E, 0 \leqslant \lambda \leqslant 1, 0 < \beta \leqslant 1 \quad \ldots(2)$$

where $g(z) = z + b_2z^2 + \ldots$ is regular and starlike of order $\alpha(0 \leqslant \alpha < 1)$ in E, that is, $\text{Re}\,\{zg'(z)/g(z)\} > \alpha$, $z \in E$. Obviously $S_\lambda(\alpha, \beta)$ is a sub-class of close-to-star functions introduced by Reade (1955). Let $K_\lambda(\alpha, \beta)$ denote the class of functions

$$F(z) = \frac{1}{z} + c_0 + c_1z + \ldots \quad \ldots(3)$$

which are regular in $0 < |z| < 1$ and satisfy the condition

$$\left|\left(\frac{F(z)}{G(z)} - 1\right)\Big/\left(\lambda\frac{F(z)}{G(z)} + 1\right)\right| < \beta, \quad z \in E, 0 \leqslant \lambda \leqslant 1, 0 < \beta \leqslant 1 \quad \ldots(4)$$

where $G(z) = \frac{1}{z} + d_0 + d_1 z + \ldots$ is regular and starlike of order α in $0 < |z| < 1$, that is, $-\mathrm{Re}\{zG'(z)/G(z)\} > \alpha$, $z \in E$.

The purpose of this paper is to study the classes $S_\lambda(\alpha, \beta)$ and $K_\lambda(\alpha, \beta)$. We obtain the coefficient estimates and radius of starlikeness and some other results for the class $S_\lambda(\alpha, \beta)$. By taking $\lambda = 0$ we get the class $S_0(\alpha, \beta)$ studied earlier by Goel (1966). Padmanabhan (1967) determined the radius of starlikeness of the class $K_0(0, 1)$. The results obtained here will generalize the results due to several earlier researchers.

2. Distortion Theorem for the Class $S_\lambda(\alpha, \beta)$

Theorem 1 — If $f \in S_\lambda(\alpha, \beta)$, then for $|z| = r$, $0 \leqslant r < 1$,

$$\frac{1-\beta r}{1+\lambda\beta r} \cdot \frac{r}{(1+r)^{2-2\alpha}} \leqslant |f(z)| \leqslant \frac{1+\beta r}{1-\lambda\beta r} \cdot \frac{r}{(1-r)^{2-2\alpha}}. \qquad \ldots(5)$$

The result is sharp.

PROOF: Since $f(z) \in S_\lambda(\alpha, \beta)$ we obtain after a simple computation

$$\frac{f(z)}{g(z)} = \frac{1-\beta w(z)}{1+\lambda\beta w(z)} \qquad \ldots(6)$$

where $w(z)$ is analytic in E and satisfies there the conditions $|w(z)| < 1$ and $w(0) = 0$.

By Schwarz's Lemma we have $|w(z)| \leqslant |z|$. This together with (6) gives

$$\frac{1-\beta r}{1+\lambda\beta r} \leqslant \left|\frac{f(z)}{g(z)}\right| \leqslant \frac{1+\beta r}{1-\lambda\beta r}, \; |z| = r. \qquad \ldots(7)$$

Since $g(z)$ is starlike of order α, we have (Pinchuk 1968)

$$\frac{r}{(1+r)^{2-2\alpha}} \leqslant |g(z)| \leqslant \frac{r}{(1-r)^{2-2\alpha}}, \; |z| = r. \qquad \ldots(8)$$

(7) together with (8) yields the inequality (5).

To show that the estimates are sharp, consider

$$f(z) = \frac{1-\beta z}{1+\lambda\beta z} g(z) \text{ where } g(z) = \frac{z}{(1-\epsilon z)^{2-2\alpha}}, \; |\epsilon| = 1.$$

It is easy to see that $f(z) \in S_\lambda(\alpha, \beta)$ and the equality signs in (5) occur when $z = \pm r$.

3. Coefficient Estimates for the Class $S_\lambda(\alpha, \beta)$

Theorem 2 — If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_\lambda(\alpha, \beta)$. Then

(*i*) $|a_2| \leqslant 2(1-\alpha) + \beta(1+\lambda)$, (*ii*) $|a_3| \leqslant (1-\alpha)(3-2\alpha) + 2(1-\alpha)(1+\lambda)\beta + \lambda(1+\lambda)\beta^2$.

The estimates are sharp

PROOF : Let

$$w(z) = \frac{1}{\beta}\left[\left(\frac{f(z)}{g(z)} - 1\right)\Big/\left(\lambda\frac{f(z)}{g(z)} + 1\right)\right]. \quad ...(9)$$

Then

$$w(z) = \sum_{m=1}^{\infty} w_m z^m$$

is analytic in E and $|w(z)| < 1$ for $z \in E$. On substituting the power series for $f(z)$, $g(z)$ and $w(z)$ in (9), we have

$$\frac{1}{\beta}\sum_{m=2}^{\infty}(a_m - b_m)z^m = \left((\lambda+1)z + \sum_{m=2}^{\infty}(\lambda a_m + b_m)z^m\right)\left(\sum_{m=1}^{\infty} w_m z^m\right). \quad ...(10)$$

Equating coefficients of z^2 and z^3 on both sides of (10), we get

$$\frac{1}{\beta}(a_2 - b_2) = (\lambda+1)w_1 \quad ...(11)$$

$$\frac{1}{\beta}(a_3 - b_3) = (\lambda+1)w_2 + (\lambda a_2 + b_2)w_1. \quad ...(12)$$

Using the fact that $|b_2| \leqslant 2 - 2\alpha$, $|b_3| \leqslant \frac{(2-2\alpha)(3-3\alpha)}{2!}$ (Robertson 1936) and $|w_1| < 1$, $|w_2| \leqslant 1 - |w_1|^2$, the inequalities (*i*) and (*ii*) easily follow from (11) and (12).

The bounds are sharp for the function $f(z)$ defined by

$$f(z) = \frac{z + \beta z^2}{(1-z)^{2-2\alpha}(1-\lambda\beta z)}.$$

Remark 1 : We have not been able to obtain sharp estimates for $|a_n|$, $n \geqslant 4$ for the class $S_\lambda(\alpha, \beta)$. However for function in the class $S_0(0, \beta)$, we determine sharp bounds for a_n in the following theorem:

Theorem 3 — If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to $S_0(0, \beta)$, then

$$|a_n| \leqslant \beta(n-1) + n, \; n \geqslant 2.$$

The bounds are sharp.

PROOF : Let

$$w(z) = \frac{1}{\beta}\left[\frac{f(z)}{g(z)} - 1\right]. \quad \ldots(13)$$

Then

$$w(z) = \sum_{m=1}^{\infty} w_m z^m \text{ is analytic in } E \text{ and } |w(z)| < 1,\ z \in E.$$

On substituting the power series for $f(z)$, $g(z)$ and $w(z)$ in (13), we have

$$\frac{1}{\beta}\sum_{m=2}^{\infty}(a_m - b_m)z^m = \left(z + \sum_{m=2}^{\infty} b_m z^m\right)\left(\sum_{m=1}^{\infty} w_m z^m\right). \quad \ldots(14)$$

Equating the coefficients of z^n on both sides of (14), we get

$$\frac{1}{\beta}(a_n - b_n) = w_{n-1} + b_2 w_{n-2} + \ldots + b_{n-1}w_1. \quad \ldots(15)$$

Now

$$w_m = \frac{1}{2\pi i}\int_{|z|=r}\frac{w(z)\,dz}{z^{m+1}},\ 0 < r < 1,\ m = 1, 2, \ldots, n-1. \quad \ldots(16)$$

From (15) and (16) we obtain

$$\frac{1}{\beta}(a_n - b_n) = \frac{1}{2\pi i}\int_{|z|=r}\frac{w(z)}{z^n}(1 + b_2 z + \ldots + b_{n-1}z^{n-2})\,dz$$

$$= \frac{1}{2\pi i}\int_{|z|=r}\frac{w(z)\,p(z)}{z^n}\,dz \quad \ldots(17)$$

where $p(z)$ is any function analytic in E with a power series expansion

$$p(z) = 1 + b_2 z + \ldots + b_{n-1}z^{n-2} + b_n z^{n-1} + \ldots \quad \ldots(18)$$

$h(z) = \sqrt{g(z^2)} = z + \sum_{m=1}^{\infty} c_{2m+1}z^{2m+1}$ is an odd univalent function because $g(z)$ is univalent.

Furthermore

$$\frac{zh'(z)}{h(z)} = \frac{zg'(z^2)}{g(z^2)}.$$

Consequently, if $g(z)$ is starlike, so is the odd function $h(z)$ and hence

$$|c_{2m+1}| \leqslant 1 \text{ for } m = 1, 2, \ldots . \quad \ldots(19)$$

Let $q(z) = 1 + c_3 z + c_5 z^2 + \ldots + c_{2n-3} z^{n-2}$, then $p(z) = (q(z))^2$ is a polynomial which satisfies (18), since $(h(z))^2 = g(z^2)$ implies

$$c_{2m-1} + c_3 c_{2m-3} + \ldots + c_{2m-3} c_3 + c_{2m-1} = b_m \text{ for } m = 1, 2, \ldots, n-1.$$

For this choice of $p(z)$ and noting that $|w(z)| < 1$ we obtain from (17)

$$\frac{1}{\beta} | a_n - b_n | \leqslant \frac{1}{2\pi} \int_0^{2\pi} \frac{| p(re^{i\theta}) |}{r^{n-1}} d\theta$$

$$= \frac{1}{2\pi r^{n-1}} \int_0^{2\pi} | q(re^{i\theta}) |^2 d\theta$$

$$= \frac{1}{r^{n-1}} (1 + | c_3 |^2 r^2 + \ldots + | c_{2n-3} |^2 r^{2n-4})$$

$$\leqslant \frac{1}{r^{n-1}} (1 + | c_3 |^2 + \ldots + | c_{2n-3} |^2).$$

Since the above inequality holds for all r in the interval $0 < r < 1$, it follows that

$$\frac{1}{\beta} | a_n - b_n | \leqslant 1 + | c_3 |^2 + \ldots + | c_{2n-3} |^2. \quad \ldots(20)$$

(20) in conjunction with (15) gives

$$\frac{1}{\beta} | a_n - b_n | \leqslant n - 1.$$

This implies that

$$| a_n | \leqslant | a_n - b_n | + | b_n |$$

$$\leqslant \beta(n-1) + n$$

since $\quad | b_n | \leqslant n$ for all n.

The bounds are sharp for the function $f(z)$ given below:

$$f(z) = \frac{z(1 + \beta z)}{(1 - z)^2}.$$

This function belongs to $S_0(0, \beta)$ and has the expansion

$$f(z) = z + (2 + \beta) z^2 + (3 + 2\beta) z^3 + \ldots + (n + \beta(n-1)) z^n + \ldots$$

4. Argument $f(z)/z$

Theorem 4 — If $f \in S_\lambda(\alpha, \beta)$ then

$$\left| \arg \frac{f(z)}{z} \right| \leqslant 2(1-\alpha)\sin^{-1} r + \sin^{-1} \frac{\beta(1+\lambda)r}{1+\lambda\beta^2 r^2}.$$

The bound is sharp.

PROOF : Since $f \in S_\lambda(\alpha, \beta)$, we have

$$\frac{f(z)}{g(z)} = \frac{1-\beta w(z)}{1+\lambda\beta w(z)}$$

where $w(z)$ is analytic in E and $|w(z)| < 1$.

An easy calculation leads to

$$\left| \frac{f(z)}{g(z)} - \frac{1+\lambda\beta^2 r^2}{1-\lambda^2\beta^2 r^2} \right| \leqslant \frac{\beta(1+\lambda)r}{1-\lambda^2\beta^2 r^2}, \quad |z| = r. \qquad ...(21)$$

Hence

$$\left| \arg \frac{f(z)}{g(z)} \right| \leqslant \sin^{-1} \frac{\beta(1+\lambda)r}{1+\lambda\beta^2 r^2}. \qquad ...(22)$$

Since g is starlike of order α, we have (Pinchuk 1968)

$$\left| \arg \frac{g(z)}{z} \right| \leqslant 2(1-\alpha)\sin^{-1} r. \qquad ...(23)$$

Therefore by (22) and (23) we get

$$\left| \arg \frac{f(z)}{z} \right| \leqslant 2(1-\alpha)\sin^{-1} r + \sin^{-1} \frac{\beta(1+\lambda)r}{1+\lambda\beta^2 r^2}.$$

To see that the result is sharp, Let

$$\frac{f(z)}{g(z)} = \frac{1-\beta\epsilon_1 z}{1+\lambda\beta\epsilon_1 z}, \; |\epsilon_1| = 1 \qquad ...(24)$$

and

$$g(z) = \frac{z}{(1+\epsilon_2 z)^{2-2\alpha}} \; |\epsilon_2| = 1. \qquad ...(25)$$

Putting

$$\epsilon_1 = \frac{r}{z}\left[\frac{\beta(1-\lambda)r}{1-\lambda\beta^2 r^2} + \frac{i\sqrt{(1-\beta^2 r^2)(1-\lambda^2\beta^2 r^2)}}{1-\lambda\beta^2 r^2}\right], \; r = |z|$$

in (24), we have

$$\arg \frac{f(z)}{g(z)} = -\sin^{-1} \frac{\beta(1+\lambda)r}{1+\lambda\beta^2 r^2}. \qquad ...(26)$$

Putting

$$\epsilon_2 = \frac{r}{z}\,[-\,r + i\sqrt{1 - r^2}] \text{ in (25), we have}$$

$$\arg \frac{g(z)}{z} = \arg \frac{1}{(1 + \epsilon_2 z)^{2-2\alpha}} = -\,(2 - 2\alpha)\sin^{-1} r. \qquad \text{...(27)}$$

From (26) and (27) we get

$$f(z) = \frac{1 - \epsilon_1 \beta z}{1 + \epsilon_1 \lambda \beta z} \cdot \frac{1}{(1 + \epsilon_2 z)^{2-2\alpha}}$$

which is the required extremal function.

5. Radius of Close-to-convexity for the Class $S_0(0, \beta)$

We shall use the following Lemma:

Lemma 1 — Let $w(z)$ be regular in E and satisfy the conditions (i) $w(0) = 0$ and (ii) $|\,w(z)\,| < 1$ for $z \in E$. Then

$$|\,zw'(z) - w(z)\,| \leqslant \frac{|\,z\,|^2 - |\,w(z)\,|^2}{1 - |\,z\,|^2}. \qquad \text{...(28)}$$

This lemma is due to Singh and Goel (1971).

Theorem 5 — Let $f \in S_0(0, \beta)$. Then f is close-to-convex for

$$|\,z\,| < \begin{cases} \dfrac{(1 - \beta) - \sqrt{\beta(1 - \beta)}}{1 - 2\beta} & \text{for } 0 < \beta \leqslant 1/2 \\[2ex] \dfrac{1}{1 + 2\beta} & \text{for } 1/2 \leqslant \beta \leqslant 1. \end{cases}$$

The result is sharp.

Proof : Since $f \in S_0(0, \beta)$, we can write

$$\frac{f(z)}{g(z)} = 1 - \beta w(z) \qquad \text{...(29)}$$

where $w(z)$ is regular in E and satisfies $w(0) = 0$ and $|\,w(z)\,| < 1$ for $z \in E$.

Differentiating (29) we get

$$\frac{zf'(z)}{g(z)} = (1 - \beta w(z))\,\frac{zg'(z)}{g(z)} - \beta z w'(z). \qquad \text{...(30)}$$

Using (28), we have

$$\text{Re}\,\frac{zf'(z)}{g(z)} \geqslant \text{Re}\left\{(1 - \beta w(z)\,\frac{zg'(z)}{g(z)}\right\} - \beta\left[\text{Re}\,w + \frac{r^2 - |\,w\,|^2}{1 - r^2}\right]. \qquad \text{...(31)}$$

Since $g(z)$ is starlike, Re $\{zg'(z)/g(z)\} > 0$ for $| z | \in E$. Hence

$$\left| \frac{zg'(z)}{g(z)} - \frac{1+r^2}{1-r^2} \right| \leqslant \frac{2r}{1-r^2}. \quad ...(32)$$

From (31) and (32) we get

$$\text{Re} \frac{zf'(z)}{g(z)} \geqslant \frac{1+r^2}{1-r^2} \text{Re} (1 - \beta w(z)) - \frac{2r}{1-r^2} | 1 - \beta w(z) |$$

$$- \beta \left[\text{Re } w(z) + \frac{r^2 - | w(z) |^2}{1-r^2} \right] = \frac{2}{1-r^2} \text{Re } p(z) - 1$$

$$- \left[\frac{\beta^2 r^2 - | 1 - p(z) |^2}{\beta(1-r^2)} \right] - \frac{2r}{1-r^2} | p(z) | \quad ...(33)$$

where $p(z) = 1 - \beta w(z)$.

It is easy to see that the transformation $p(z) = 1 - \beta w(z)$ maps the circle $| w(z) | \leqslant r$ onto the circle

$$| p(z) - 1 | \leqslant \beta r, \; r = | z | . \quad ...(34)$$

If we put $p(z) = R\, e^{i\theta}$ and denote the right-hand side of (33) by $S(R, \theta)$, then

$$S(R, \theta) = \frac{2(\beta - 1)}{(1-r^2)\beta} R \cos \theta - \frac{2r\, R}{1-r^2} - \left[\frac{\beta^2 r^2 - (1 + R^2)}{\beta(1-r^2)} \right] - 1. \quad ...(35)$$

Now

$$\frac{\partial S}{\partial \theta} = \frac{2(1-\beta)}{(1-r^2)\beta} R \sin \theta \text{ where } 1 - \beta r \leqslant R \leqslant 1 + \beta r.$$

Therefore, the minimum of $S(R, \theta)$ inside the circle $| p(z) - 1 | \leqslant \beta r$ is attained for $\theta = 0$. By putting $\theta = 0$ in (35) we obtain

$$S(R, 0) = L(R) = \frac{(1-\beta)(1 + \beta r^2)}{\beta(1-r^2)} - 2 \frac{1 - \beta + \beta r}{\beta(1-r^2)} R + \frac{1}{\beta(1-r^2)} R^2$$

$$\frac{\partial L}{\partial R} = - 2 \frac{(1 - \beta + \beta r)}{\beta(1-r^2)} + \frac{2}{\beta(1-r^2)} R.$$

We see that the absolute minimum of $L(R)$ in $(0, \infty)$ is attained at $R_0 = 1 - \beta + \beta r$ and equals

$$\frac{1 - \beta - 2(1-\beta)\, r + (1 - 2\beta)\, r^2}{1-r^2}. \quad ...(36)$$

It is clear that $R_0 < 1 + \beta r$, but R_0 is not always greater than $1 - \beta r$. In such a case when $R_0 \notin [1 - \beta r, 1 + \beta r]$ the minimum of $L(R)$ on the segment $[1 - \beta r, 1 + \beta r]$ is attained at $R_1 = 1 - \beta r$ and equals

$$\frac{1 - 2(1 + \beta) r + (1 + 2\beta) r^2}{\beta(1 - r^2)}. \quad \ldots(37)$$

The two values given by (36) and (37) coincide for such values of β for which $R_0 = R_1$.

We thus conclude that

$$\operatorname{Re} \frac{zf'(z)}{g(z)} \geqslant \frac{1 - \beta - 2(1 - \beta) r + (1 - 2\beta) r^2}{1 - r^2} \text{ for } R_0 \geqslant R_1 \quad \ldots(38)$$

and

$$\operatorname{Re} \frac{zf'(z)}{g(z)} \geqslant \frac{1 - 2(1 + \beta) r + (1 + 2\beta) r^2}{\beta(1 - r^2)} \text{ for } R_0 \leqslant R_1. \quad \ldots(39)$$

The equality signs in (39) and (38) are attained respectively for the functions

$$f_1(z) = \frac{z}{(1 - z)^2} (1 - \beta z) \quad \ldots(40)$$

and

$$f_2(z) = \frac{z}{(1 - z)^2} \left(1 - \beta \frac{z(z - q)}{1 - qz}\right), \; g(z) = \frac{z}{(1 - z)^2} \quad \ldots(41)$$

where q is determined from $\operatorname{Re} p_2(r) = R_0$, where $p_2(z) = \dfrac{f_2(z)}{g(z)}$.

Therefore the radius of close-to-convexity for the class $S_0(0, \beta)$ is given by

$$1 - \beta - 2(1 - \beta) r + (1 - \beta) r^2 = 0 \text{ for } R_0 \geqslant R_1 \quad \ldots(42)$$

and

$$1 - 2(1 + \beta) r + (1 + 2\beta) r^2 = 0 \text{ for } R_0 \leqslant R_1. \quad \ldots(43)$$

Eliminating r from (43) and $R_0 = R_1$, we get $\beta = 1/2$.

So two values of radius of close-to-convexity given by (42) and (43) are equal when $\beta = 1/2$.

Hence f is close-to-convex for

$$|z| < \begin{cases} \dfrac{(1 - \beta) - \sqrt{\beta(1 - \beta)}}{1 - 2\beta} & \text{for } 0 < \beta \leqslant 1 \\[2ex] \dfrac{1}{1 + 2\beta} & \text{for } 1/2 \leqslant \beta \leqslant 1. \end{cases}$$

This proves the theorem.

6. Radius of Starlikeness for the Class $S_\lambda(\alpha, \beta)$

The following Lemmas will be used.

Lemma 2 — Let $w(z)$ be regular in E and satisfy the conditions (i) $w(0) = 0$ and (ii) $|w(z)| < 1$ for $z \in E$. Then for $0 < \beta \leqslant 1$, $0 \leqslant \lambda \leqslant 1$, we have

$$\operatorname{Re}\left\{\frac{zw'(z)}{(1-\beta w(z))(1+\lambda\beta w(z))}\right\} \leqslant -\frac{1}{\beta^2(1+\lambda)^2}\left[\operatorname{Re}\left\{\lambda\beta p(z) - \frac{\beta}{p(z)} + \beta(1+\lambda)\right\} - \frac{r^2|\lambda\beta p(z)+\beta|^2 - |1-p(z)|^2}{(1-r^2)|p(z)|}\right] \quad \ldots(44)$$

$$\operatorname{Re}\left\{\frac{zw'(z)}{(1-\beta w(z))(1+\lambda\beta w(z))}\right\} \geqslant \frac{1}{\beta^2(1+\lambda)^2}\left[\operatorname{Re}\left\{\beta(\lambda-1) + \frac{\beta}{p(z)} - \lambda\beta p(z)\right\} - \frac{r^2|\lambda\beta p(z)+\beta|^2 - |1-p(z)|^2}{(1-r^2)|p(z)|}\right] \quad \ldots(45)$$

where $p(z) = \dfrac{1-\beta w(z)}{1+\lambda\beta w(z)}$, $r = |z|$.

(44) and (45) follow easily from (28).

Lemma 3 — Let $p(z) = \dfrac{1-\beta w(z)}{1+\lambda\beta w(z)}$, then for $|z| = r$, $0 \leqslant r < 1$, we have

$$\operatorname{Re}\left[\lambda\beta p(z) - \frac{\beta}{p(z)}\right] - \left(\frac{r^2|\lambda\beta p(z)+\beta|^2 - |1-p(z)|^2}{(1-r^2)|p(z)|}\right.$$

$$\geqslant \begin{cases} \dfrac{\lambda\beta(1-\beta r)}{1+\lambda\beta r} - \dfrac{\beta(1+\lambda\beta r)}{1-\beta r} & \text{for } R_0 \leqslant R_1, \\[2ex] \dfrac{2}{1-r^2}\left[\sqrt{(1-\beta)(1+\lambda\beta)[1+(1-\lambda)\beta r^2 - \lambda\beta^2 r^4]} - (1+\lambda\beta^2 r^2)\right] & \\ \qquad \text{for } R_0 \geqslant R_1 & \end{cases} \quad \ldots(46)$$

and

$$\operatorname{Re}\left[\lambda\beta p(z) - \frac{\beta}{p(z)}\right] + \left(\frac{r^2|\lambda\beta p(z)+\beta|^2 - |1-p(z)|^2}{(1-r^2)|p(z)|}\right.$$

$$\leqslant \begin{cases} \dfrac{2}{1-r^2}\left[(1+\lambda\beta^2 r^2) - \sqrt{(1+\beta)(1-\lambda\beta)[1-\beta(1-\lambda)r^2 - \lambda\beta^2 r^4]}\right] & \\ \qquad \text{for } R^* \leqslant R_2, & \\[2ex] \dfrac{-\beta[\lambda(1-\lambda)\beta^2 r^4 + 4\lambda\beta r^3 - (1-\lambda)(1+\lambda\beta^2)r^2 - 4\lambda\beta r + 1 - \lambda]}{(1-r^2)(1+\beta r)(1-\lambda\beta r)} & \\ \qquad \text{for } R^* \geqslant R_2, & \end{cases} \quad \ldots(47)$$

where

$$R_0 = \sqrt{\frac{(1-\beta)(1+\beta r^2)}{(1+\lambda\beta)(1-\lambda\beta r^2)}}, \quad R^* = \sqrt{\frac{(1+\beta)(1-\beta r^2)}{(1-\lambda\beta)(1+\lambda\beta r^2)}}$$

$$R_1 = \frac{1-\beta r}{1+\lambda\beta r} \text{ and } R_2 = \frac{1+\beta r}{1-\lambda\beta r}.$$

These bounds are sharp.

PROOF : It is easy to see that the transformation $p(z) = \dfrac{1-\beta w(z)}{1+\lambda\beta w(z)}$ maps the circle $|w(z)| \leqslant r$ onto the circle $|p(z) - a| \leqslant d$ where

$$a = \frac{1+\lambda\beta^2 r^2}{1-\lambda^2\beta^2 r^2} \text{ and } d = \frac{\beta(1+\lambda)r}{1-\lambda^2\beta^2 r^2}.$$

Putting $p(z) = a + u + iv$, $R^2 = (a+u)^2 + v^2$ and denoting the left-hand side of (46) by $S(u, v)$, we have

$$\left.\begin{aligned} S(u,v) &= \lambda\beta(a+u) - \frac{\beta(a+u)}{R^2} + \frac{(1-\lambda^2\beta^2)(u^2+v^2-d^2)}{(1-r^2)R} \\ \frac{\partial S}{\partial v} &= vR^{-4}T(u,v) \end{aligned}\right\} \quad \ldots(48)$$

where $T(u, v) = 2\beta(a+u) + \dfrac{2(1-\lambda^2\beta^2 r^2)}{1-r^2} R^3 + \dfrac{(1-\lambda^2\beta^2 r^2)(d^2-u^2-v^2)R}{1-r^2}.$

Clearly $T(u,v) \geqslant 2(a+u)\left(\left(\dfrac{1-\lambda^2\beta^2 r^2}{1-r^2}\right)(a-d)^2 + \beta\right) \geqslant 0.$

Hence the minimum of $S(u, v)$ inside the circle $|p(z) - a| \leqslant d$ is attained on the diameter $v = 0$. On putting $v = 0$ in (48) we have

$$L(R) \equiv S(u,0) = \lambda\beta R - \frac{\beta}{R} + \frac{1-\lambda^2\beta^2}{(1-r^2)R}[R^2 + a^2 - 2aR - d^2]$$

where $R = a + u$ and $a - d \leqslant R \leqslant a + d$.

The absolute minimum of $L(R)$ in $(0, \infty)$ is attained at

$$R_0 = \sqrt{\frac{(1-\beta)(1+\beta r^2)}{(1+\lambda\beta)(1-\lambda\beta r^2)}} \quad \ldots(49)$$

and equals

$$L(R_0) = \frac{2}{1-r^2}[\sqrt{(1-\beta)(1+\lambda\beta)[1+(1-\lambda)\beta r^2 - \lambda\beta^2 r^4]} - (1+\lambda\beta^2 r^2)]. \quad \ldots(50)$$

It is easy to see that $R_0 < a + d$, but R_0 is not always greater than $a - d$. In such a case when $R_0 \notin [a-d, a+d]$ the minimum of $L(R)$ on the segment $[a-d, a+d]$ is attained at $R_1 = a - d$ and equals

$$L(R_1) = L(a-d) = \frac{\lambda\beta(1-\beta r)}{1+\lambda\beta r} - \frac{\beta(1+\lambda\beta r)}{(1-\beta r)}. \quad \ldots(51)$$

$L(R_0) = L(R_1)$ for such values of λ and β for which $R_0 = R_1$. The inequalities (46) follow from (51) and (50).

Proceeding as above the inequalities (47) can be proved easily. We now show that the bounds are sharp.

The inequalities (46) are sharp for the functions

$$p_1(z) = \frac{1 - \beta z}{1 + \lambda\beta z} \text{ and } p_2(z) = \frac{1 - \beta w_2(z)}{1 + \lambda\beta w_2(z)}$$

where

$$w_2(z) = \frac{z(z - \nu)}{1 - \nu z} \text{ and } \nu \text{ is determined from the condition}$$

$$| p_2(z) | = R_0 = \operatorname{Re} p_2(r).$$

The inequalities (47) are sharp for the functions

$$p_3(z) = \frac{1 - \beta w_3(z)}{1 + \lambda\beta w_3(z)}$$

where $w_3(z) = \dfrac{-z(z - q)}{1 - qz}$ and q is determined from the condition

$$| p_3(z) | = R^* = \operatorname{Re} p_3(r)$$

and

$$p_4(z) = \frac{1 + \beta z}{1 - \lambda\beta z}.$$

Theorem 6 — If $f \in S_\lambda(\alpha, \beta)$, then

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geqslant \begin{cases} M_1(r) \text{ for } R_0 \leqslant R_1 \\ M_2(r) \text{ for } R_0 \geqslant R_1 \end{cases} \qquad \ldots(52)$$

where

$$M_1(r) = \frac{1 + [2(\alpha - \beta) - 1]\, r - \beta\, [\lambda\beta + (2\alpha - 1)(1 - \lambda) + 1 + \lambda]\, r^2 - (2\alpha - 1)\, \lambda\beta^2 r^3}{(1 + r)(1 - \beta r)(1 + \lambda\beta r)},$$

$$M_2(r) = \frac{2[(1-\beta)(1+\lambda\beta)(1+(1-\lambda)\beta r^2 - \beta^2 r^4)]^{1/2} - 2[1 - \beta + \beta(1-\alpha)(1+\lambda)\, r - \beta(\lambda(1-\alpha+\beta) - \alpha)\, r^2]}{\beta(1+\lambda)(1-r^2)}$$

and R_0, R_1 are as in Lemma 3.

The bounds are sharp.

PROOF : $f \in S_\lambda(\alpha, \beta)$ implies

$$\frac{f(z)}{g(z)} = \frac{1 - \beta w(z)}{1 + \lambda\beta w(z)} \qquad \ldots(53)$$

where $w(z)$ is regular in E and $|w(z)| < 1$.

Differentiating (53) logarithmically, we have

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} - \frac{\beta(1 + \lambda)\, zw'(z)}{(1 + \lambda\beta w(z))\,(1 - \beta w(z))}. \qquad \ldots(54)$$

Applying (28), (54) yields

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geqslant \operatorname{Re}\frac{zg'(z)}{g(z)} + \frac{1}{\beta(1 + \lambda)}\left[\operatorname{Re}\left\{\lambda\beta p(z) - \frac{\beta}{p(z)} + \beta(1 - \lambda)\right\} - \frac{r^2\,|\lambda\beta p(z) + \beta|^2 - |1 - p(z)|^2}{(1 - r^2)\,|p(z)|}\right] \qquad \ldots(55)$$

where $p(z) = \dfrac{1 - \beta w(z)}{1 + \lambda\beta w(z)}$.

Since g is starlike of order α, therefore we have

$$\operatorname{Re}\frac{zg'(z)}{g(z)} \geqslant \frac{1 + (2\alpha - 1)\,r}{1 + r}, \; |z| = r. \qquad \ldots(56)$$

Using (46), (55) and (56) give the required inequalities.

To show that the bounds are sharp, choose $g_1(z)$, starlike of order α such that

$$\frac{zg_1'(z)}{g_1(z)} = \frac{1 + (2\alpha - 1)\,w_1(z)}{1 + w_1(z)}$$

and take $f_1(z)$ such that it satisfies

$$p_1(z) = \frac{f_1(z)}{g_1(z)} = \frac{1 - \beta w_1(z)}{1 + \lambda\beta w_1(z)}.$$

From the proof of Lemma 3, it is clear that the bounds $M_1(r)$ and $M_2(r)$ are attained when $R = a - d$ and $R = R_0$ respectively, where $R = |p(z)| = \operatorname{Re} p(z)$. Since $a - d = \dfrac{1 - \beta r}{1 + \lambda\beta r}$, the function $w_1(z) = z$ will give $g_1(z)$ and consequently $f_1(z)$ such that $|p_1(z)| = \operatorname{Re} p_1(z) = a - d$ at $z = r$, for all r satisfying $R_0 \leqslant R_1$. When $R_0 \geqslant R_1$ we choose $w_2(z) = \dfrac{z(z - \nu)}{1 - \nu z}$ where ν is determined by the condition that

$$p_2(z) = \frac{1 - \beta w_2(z)}{1 + \lambda\beta w_2(z)} \text{ satisfies } |p_2(z)| = R_0 = \operatorname{Re} p_2(z) \text{ at } z = r.$$

Since we have $a - d \leqslant R_0 \leqslant a + d$

therefore

$$\frac{1 - \beta r}{1 + \lambda\beta r} \leqslant \frac{1 - \beta t}{1 + \lambda\beta t} \leqslant \frac{1 + \beta r}{1 - \lambda\beta r}$$

where $t = w_2(r)$ and this implies

$$\frac{r^2(r - \nu)^2}{(1 - \nu r)^2} = t^2 \leqslant r^2$$

and hence $|\nu| \leqslant 1$. So taking $g_2(z)$, defined by

$$\frac{zg_2'(z)}{g_2(z)} = \frac{1 + (2\alpha - 1)w_2(z)}{1 + w_2(z)} \text{ and } f_2(z) = g_2(z) \cdot \frac{1 - \beta w_2(z)}{1 + \lambda\beta w_2(z)}$$

which satisfies $|p_2(z)| = \text{Re}\, p_2(r) = R_0$, where

$$p_2(z) = \frac{f_2(z)}{g_2(z)} = \frac{1 - \beta w_2(z)}{1 + \lambda\beta w_2(z)}.$$

The bound $M_2(r)$ is attained at $z = r$.

Theorem 7 — If $f \in S_\lambda(\alpha, \beta)$, then f is starlike in

$$|z| < \begin{cases} r_1 \text{ for } R_0 \leqslant R_1 \\ r_2 \text{ for } R_0 \geqslant R_1 \end{cases}$$

where R_0 and R_1 are as in Lemma 3 and r_1 and r_2 are respectively the smallest positive roots of the following equations:

$$1 + [2(\alpha - \beta) - 1]\, r - \beta\, [\lambda\beta + (2\alpha - 1)(1 - \lambda) + 1 + \lambda]\, r^2 - (2\alpha - 1)\, \lambda\beta^2 r^3 = 0 \qquad \ldots(57)$$

$$\beta - 1 + 2(1 - \alpha)(1 - \beta)\, r + [\beta(1 + \lambda)(1 - \alpha)^2 + (1 - \beta) \times (2\alpha - 1 + \lambda\beta)] r^2 - 2\beta(1 - \alpha)\, [\lambda(1 - \beta) - \alpha(1 + \lambda)]\, r^3 + \beta\, [\alpha^2(1 + \lambda) - \lambda(1 - \beta\,(2\alpha - 1)]\, r^4 = 0. \qquad \ldots(58)$$

The result is sharp.

This theorem follows easily from Theorem 6.

Remark 1 : In the above theorem, the two values r_1 and r_2 become equal when $R_0 = R_1$. In the particular case when $\lambda = 0$, eliminating r from (57) and $R_0 = R_1$, we get $\beta = \frac{(2 - \alpha^2) + \alpha(4 + \alpha^2)^{1/2}}{4}$. So it easily follows that f is starlike for $|z| < \frac{[(2\beta - 2\alpha + 1)^2 + 8\alpha\beta]^{1/2} - (2\beta - 2\alpha + 1)}{4\alpha\beta}$ when $\beta \geqslant \frac{(2 - \alpha^2) + \alpha(4 + \alpha^2)^{1/2}}{4}$.

For $\beta \leqslant \frac{(2-\alpha^2)+\alpha(4+\alpha^2)^{1/2}}{4}$, f is starlike for $|z| < r_2$ where r_2 is the smallest positive root of the equation

$$\alpha^2\beta r^4 + 2\alpha\beta(1-\alpha)r^3 + [2\alpha + 2\beta - 1 - 4\alpha\beta + \alpha^2\beta]r^2 + 2(1-\alpha)(1-\beta)r + \beta - 1 = 0.$$

This result was obtained by Goel (1966). Further taking $\beta = 1$ and $\alpha = 0$, we see that f is starlike for $|z| < 1/3$. This is a result due to Mac-Gregor (1963).

7. Radius of Starlikeness for the Class $K_\lambda(\alpha, \beta)$

Theorem 8 — If $F \in K_\lambda(\alpha, \beta)$, then for $|z| = r < 1$,

$$-\operatorname{Re}\frac{zF'(z)}{F(z)} \geqslant \begin{cases} P_1(r) \text{ for } R^* \leqslant R_2 \\ P_2(r) \text{ for } R^* \geqslant R_2 \end{cases} \qquad \ldots(59)$$

where

$$P_1(r) = \frac{1+(2\alpha-1)r}{1+r} - \frac{1-\lambda}{1+\lambda} - \frac{2}{\beta(1+\lambda)(1-r^2)} \times [(1+\lambda\beta^2r^2) - \sqrt{(1+\beta)(1-\lambda\beta)\{1-\beta(1-\lambda)r^2-\lambda\beta^2r^4\}}]$$

$$P_2(r) = \frac{1+(2\alpha-1)r}{1+r} - \frac{1-\lambda}{1+\lambda} + \frac{1-\lambda-4\lambda\beta r-(1-\lambda)(1+\lambda\beta^2)r^2+4\lambda\beta r^3+\lambda(1-\lambda)\beta^2r^4)}{(1+\lambda)(1-r^2)(1-\lambda\beta r)(1+\beta r)}$$

and R^*, R_2 are as given in Lemma 3.

These estimates are sharp.

Proof : Since $F \in K_\lambda(\alpha, \beta)$, therefore we can write

$$\frac{F(z)}{G(z)} = \frac{1-\beta w(z)}{1+\lambda\beta w(z)} \qquad \ldots(60)$$

where $w(z)$ is analytic in E and $|w(z)| < 1$.

Differentiating (60) logarithmically, we have

$$-\frac{zF'(z)}{F(z)} = -\frac{zG'(z)}{G(z)} + \frac{\beta(1+\lambda)zw'(z)}{(1-\beta w(z)(1+\lambda\beta w(z))}. \qquad \ldots(61)$$

Applying (45), (61) yields

$$-\operatorname{Re}\frac{zF'(z)}{F(z)} \geqslant -\operatorname{Re}\frac{zG'(z)}{G(z)} + \frac{1}{\beta(1+\lambda)} \times$$

(*equation continued on p.* 1323)

$$\times\left[\operatorname{Re}\left\{\beta(\lambda-1)+\frac{\beta}{p(z)}-\lambda\beta p(z)\right\}-\frac{r^2\,|\,\lambda\beta p(z)+\beta\,|^2-|\,1-p(z)\,|^2}{(1-r^2)\,|\,p(z)\,|}\right] \quad \ldots(62)$$

where $p(z)=\dfrac{1-\beta w(z)}{1+\lambda\beta w(z)}$.

Since G is starlike of order α, therefore we have

$$-\operatorname{Re}\frac{zG'(z)}{G(z)}\geqslant\frac{1+(2\alpha-1)r}{1+r},\ |\,z\,|=r. \quad \ldots(63)$$

Using (47), (62) and (63) gives the required inequalities. To see that the bounds are sharp, choose $G_1(z)$, starlike of order α such that

$$-\frac{zG_1'(z)}{G_1(z)}=\frac{1+(2\alpha-1)w_1(z)}{1+w_1(z)}$$

and take $F_1(z)$ such that it satisfies

$$p_1(z)=\frac{F_1(z)}{G_1(z)}=\frac{1-\beta w_1(z)}{1+\lambda\beta w_1(z)}$$

where $w_1(z)=-\dfrac{z(z-q)}{1-qz}$ with q determined by the condition $\operatorname{Re}p_1(r)=R^*$. The bound $P_1(r)$ is sharp for the function $F_1(z)$ at $z=r$. The bound $P_2(r)$ is sharp for the function

$$F_2(z)=\frac{z}{(1-z)^{2-2\alpha}}\,\frac{1+\beta z}{1-\lambda\beta z}\text{ at } z=r.$$

Theorem 9 — If $F\in K_\lambda(\alpha,\beta)$, then for $|\,z\,|=r<1$, F is starlike in

(*i*) $0<|\,z\,|<r_1$ for $R^*\leqslant R_2$

(*ii*) $0<|\,z\,|<r_2$ for $R^*\leqslant R_2$

where r_1 and r_2 are the smallest positive roots of the following equations respectively:

$$\begin{aligned}&(1+\lambda)(1-\lambda\beta)-2(1-\lambda\beta)(1-\alpha)(1+\lambda)r-[\beta(1-\alpha)^2(1+\lambda)^2\\&\quad+(1-\lambda\beta)(\beta+(2\alpha-1)(1+\lambda))]\,r^2-2\beta(1-\alpha)(1+\lambda)\\&\quad\times[\alpha(1+\lambda)-(1-\lambda\beta)]\,r^3-\beta[(1-\alpha-\lambda(a+\beta))^2\\&\quad+\lambda(1+\beta)(1-\lambda)]\,r^4=0\end{aligned} \quad \ldots(64)$$

$$\begin{aligned}&1+2(\alpha-1-\lambda\beta)\,r+[(1-\lambda)\,\beta(2\alpha-2)-(2\alpha-1)-\lambda\beta^2]\,r^2\\&\quad+2\beta\,[1-\alpha(1-\lambda)]\,r^3+(2\alpha-1)\,\lambda\beta^2r^4=0.\end{aligned} \quad \ldots(65)$$

The result is sharp.

This theorem follows easily from Theorem 8.

Remark 2 : By putting $\lambda = \alpha = 0$ and $\beta = 1$ in (64) it can be easily seen that $F(z)$ is starlike for $0 < | z | < \sigma$, where $\sigma = \frac{1}{2}[((2\sqrt{2} - 1))^{1/2} - (\sqrt{2} - 1)]$. This result is due to Padmanabhan (1967).

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