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OPERATIONAL DERIVATION OF GENERATING RELATIONS OF ORTHOGONAL POLYNOMIALS RELATED TO ULTRA-SPHERICAL AND HERMITE POLYNOMIALS

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Some generating relations for generalized even and odd ultraspherical polynomials and generalized even and odd Hermite polynomials have been deduced from the relationships (1.5) – (1.8) of these polynomials with Jacobi and Laguerre polynomials. Operational derivation of one of these generating relations has also been outlined.

1. INTRODUCTION

Recently, Thakare and Bhonsle (1974) studied orthogonal polynomials related to the ultraspherical polynomials and Thakare and Karande (1973) studied orthogonal polynomials related to the Hermite polynomials. For these polynomials, they obtained recurrence relations, hypergeometric forms, some generating functions and Rodrigues' formulae. We enlist the Rodrigues' formulae as

$$T_{2pk}(z) \equiv T_{2pk}(z, \alpha, \beta) = \frac{(-1)^p z^\beta}{(1+\alpha)_p} (1-z)^{-\alpha} D^p [z^{p-\beta}(1-z)^{\alpha+p}] \quad \ldots(1.1)$$

$$T_{2pk+1}(z) \equiv T_{2pk+1}(z, \alpha, \beta) = \frac{(-1)^p (1-z)^{-\alpha}}{(1+\alpha)_p} D^p [z^{p+\beta}(1-z)^{\alpha+p}], \quad \ldots(1.2)$$

where $\beta = 1/2k$.

$$H_{2pk}(z) \equiv H_{2pk}(z, \beta) = \frac{(-1)^p (1+p)_p}{(1+\beta)_p} z^{-\beta} e^z D^p [e^{-z} z^{\beta+p}] \quad \ldots(1.3)$$

and

$$H_{2pk+1}(z) \equiv H_{2pk+1}(z, \beta) = \frac{(-1)^p 2^{2p+1} (3/2)_p}{(1-\beta)_p} e^z D^p [e^{-z} z^{p-\beta}] \quad \ldots(1.4)$$

where $\beta = -1/2k$.

It is easy to observe that the so-called even and odd generalized ultraspherical polynomials are special cases of Jacobi polynomials. Furthermore, the so-called even and odd generalized Hermite polynomials are essentially the Laguerre polynomials.

Specifically, we have the following connecting relationships

$$T_{2pk}(z, \alpha, \beta) = (p!/(1 + \alpha)_p)\, P_p^{(\alpha,-\beta)}\,(2z - 1) \qquad ...(1.5)$$

$$T_{2pk+1}(z, \alpha, \beta) = (p!/(1 + \alpha)_p)\, z^{\beta} P_p^{(\alpha,\beta)}\,(2z - 1) \qquad ...(1.6)$$

where $\beta = 1/2k$;

and

$$H_{2pk}(z, \beta) = (-1)^p\, 2^{2p}(\tfrac{1}{2})_p\, \{(1 + \beta)_p\}^{-1}\, L_p^{(\beta)}\,(z) \qquad ...(1.7)$$

$$H_{2pk+1}(z, \beta) = (-1)^p\, 2^{2p+1}(\tfrac{3}{2})_p\, \{(1 - \beta)_p\}^{-1}\, z^{-\beta} L_p^{(-\beta)}\,(z) \qquad ...(1.8)$$

where $\beta = -1/2k$.

In view of relationships (1.5) - (1.8) various results for these polynomials [including those given by Thakare and Bhonsle (1974) and Thakare and Karande (1973)] can be derived from the corresponding well-known results involving Jacobi and Laguerre polynomials. [See also Srivastava and Singh (1980)].

The object of the present note is to derive some generating relations for above-mentioned polynomials by using (1.5) - (1.8). Finally, we outline operational derivation of one of our results without using the relationship (1.5).

2. Generating Relations

The well-known result involving special Jacobi polynomials [see, e.g., Srivastava (1969), p. 593, eqn. (18)]:

$$\sum_{n=0}^{\infty} P_n^{(\alpha-n,\beta-n)}\,(x)\, t^n = (1 + \tfrac{1}{2}(1 + x)\, t)^{\alpha}\, (1 + \tfrac{1}{2}(x - 1)\, t)^{\beta}$$

can be translated in terms of the even and odd generalized ultraspherical polynomials in the forms

$$\sum_{p=0}^{\infty} (-\alpha)_p\, T_{2pk}(z, \alpha - p, \beta + p)\, \frac{t^p}{p!} = (1 - zt)^{\alpha}\, (1 + t - zt)^{-\beta} \qquad ...(1.9)$$

and

$$\sum_{p=0}^{\infty} (-\alpha)_p\, T_{2pk+1}(z, \alpha - p, \beta - p)\, \frac{t^p}{p!} = (1 - t)^{\alpha}\, (z + t - zt)^{\beta}. \qquad ...(1.10)$$

Similarly, from (1.7), (1.8) and the well-known result

$$\sum_{n=0}^{\infty} L_n^{(\alpha-n)}(x)\, t^n = e^{-xt}(1+t)^{\alpha},\ |t| < 1$$

we obtain the following formulae for even and odd generalized Hermite polynomials

$$\sum_{p=0}^{\infty} \psi_1(p)\, z^{-p} H_{2pk}(z, \beta - p) \frac{t^p}{p!} = e^{-t}\left(1 + \frac{t}{z}\right)^{\beta} \qquad \text{...(1.11)}$$

$$\sum_{p=0}^{\infty} \psi_2(p)\, H_{2pk+1}(z, p - \beta) \frac{t^p}{p!} = e^{-t}(z+t)^{\beta} \qquad \text{...(1.12)}$$

$$\sum_{p=0}^{\infty} \psi_1(p)\, H_{2pk}(z, \beta - p) \frac{t^p}{p!} = e^{-tz}(1+t)^{\beta} \qquad \text{...(1.13)}$$

and

$$\sum_{p=0}^{\infty} \psi_2(p)\, H_{2pk+1}(z, p - \beta) \frac{t^p}{p!} = e^{-t} z^{\beta}\left(1 + \frac{t}{z}\right)^{\beta} \qquad \text{...(1.14)}$$

where

$$\psi_1(p) = \frac{(1+\beta-p)_p}{(-1)^p (1+p)_p},\quad \psi_2(p) = \frac{(1+\beta-p)_p}{(-1)^p\, 2^{2p+1}(3/2)_p}.$$

The above formulae can also be obtained by using operational technique. For example, consider

$$\sum_{p=0}^{\infty} \phi(p)\, z^{-p}(1-z)^{-p}\, T_{2pk}(z, \alpha - p, \beta + p) \frac{t^p}{p!}$$

$$= z^{\beta}(1-z)^{-\alpha} \exp(tD)\, (z^{-\beta}(1-z)^{\alpha}) \qquad \text{(by (1.1))}$$

$$= z^{\beta}(1-z)^{-\alpha}\, (z+t)^{-\beta}\, (1-(z+t))^{\alpha}.$$

Thus we have

$$\sum_{p=0}^{\infty} \phi(p)\, (z(1-z))^{-p}\, T_{2pk}(z, \alpha - p, \beta + p) \frac{t^p}{p!}$$

$$= \left(1 + \frac{t}{z}\right)^{-\beta}\left(1 - \frac{t}{1-z}\right)^{\alpha} \qquad \text{...(1.15)}$$

where

$$\phi(p) = \frac{(1 + \alpha - p)_p}{(-1)_p} = (-\alpha)_p.$$

Again the relation (1.1), in view of Taylor's formula

$$\sum_{n=0}^{\infty} \frac{(xt)^n}{n!} D^n f(x) = f(x(1 + t))$$

yields

$$\sum_{p=0}^{\infty} \phi(p)\,(1 - z)^{-p}\, T_{2pk}(z, \alpha - p, \beta + p)\,\frac{t^p}{p!}$$

$$= (1 + t)^{-\beta}\left(1 - \frac{t}{1 - z}\right)^{\alpha}. \qquad \text{...(1.16)}$$

By replacing t in (1.15) by $\{z(1 - z)\,t\}$ and in (1.16) by $\{(1 - z)\,t\}$, we obtain the equivalent form (1.9). Similarly, other results can be derived.

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