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ON THE RATE OF CONVERGENCE OF THE HERMITE-FEJÉR PROCESS ON THE TCHEBYCHEFF MATRIX OF THE SECOND KIND

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In the present paper the estimate for rate of convergence of the sequence $H_n(f, x)$ is obtained.

§1. Let $f(x)$ be a continuous function defined on the closed interval $[-1, 1]$ and

$$U_{n-2}(x) = \frac{\sin (n-1)\theta}{\sin \theta}, \ \cos\theta = x$$

be the $(n-2)$th Tchebycheff polynomial of second kind. Let

$$x_{kn} = \cos\frac{(k-1)\pi}{n-1}, \ k = 1, ..., n \qquad ...(1.1)$$

be the zeros of $(1-x^2)\,U_{n-2}(x)$. Then the Hermite-Fejér interpolation polynomial $H_n(f, x)$ of degree $\leqslant 2n-1$ constructed on the nodes x_{kn} is given by

$$H_n(f, x) = f(1)\left[1 + \frac{2n^2-4n+3}{3}(1-x)\right]\left[\frac{1+x}{2}\frac{U_{n-2}(x)}{n-1}\right]^2$$
$$+ f(-1)\left[1 + \frac{2n^2-4n+3}{3}(1+x)\right]\left[\frac{1-x}{2}\frac{U_{n-2}(x)}{n-1}\right]^2$$
$$+ \sum_{k=2}^{n-1} f(x_{kn})\left[1 + \frac{x_{kn}}{1-x_{kn}}(x-x_{kn})\right]\left[\frac{(1-x^2)\,U_{n-2}(x)}{(n-1)(x-x_{kn})}\right]^2. \qquad ...(1.2)$$

Saxena (1974) has shown that the sequence of these polynomials $H_n(f, x)$ converges uniformly to the given continuous function $f(x)$ in the interval $[-1, 1]$ as n tends to infinity. The aim of the present paper is to obtain the estimate for rate of convergence of the sequence $H_n(f, x)$.

Let us denote by $C_\omega[-1, 1]$ the class of all those functions for which

$$\omega(f; t) \leqslant C_1\omega(t)$$

where $\omega(f; t)$ is the modulus of continuity of $f(x)$, $\omega(t)$ is a certain modulus of continuity and C_1 (later on C_2, C_3, ...) some positive constant. We shall prove

Theorem 1 — If $f(x) \in C_\omega[-1, 1]$, then for $x \in [-1, 1]$, we have

$$| H_n(f, x) - f(x) | = O(1) \sum_{i=1}^{n} \frac{1}{i^2} \omega \left[\frac{i(1 - x^2)^{1/2} \, | U_{n-2}(x) |}{n} \right]. \qquad \ldots(1.3)$$

The result appears to be interesting in the sense that $f(x)$ can be approximated arbitrarily at the nodes (1.1) by $H_n(f, x)$. Further, if $f(x)$ belongs to Lipschitz class of order α i.e. $\omega(t) \leqslant t^\alpha$, we get from (1.3)

$$| H_n(f, x) - f(x) | \leqslant C_2 \frac{(1 - x^2)^{1/2} \, | U_{n-2}(x) |}{n^\alpha} \quad \text{if } 0 < \alpha < 1$$

and $$| H_n(f, x) - f(x) | \leqslant C_3 \frac{(1 - x^2)^{1/2} \, | U_{n-2}(x) | \log n}{n} \quad \text{if } \alpha = 1.$$

Evidently, our theorem is the best possible for $f(x) \in \text{Lip}\, \alpha \; (0 < \alpha < 1)$ when $x \in [-1, 1]$. We shall also show that our estimation is precise for

$$f(x) \in \text{Lip}\, 1 (-1 \leqslant x \leqslant 1)$$

by proving the following:

Theorem 2 — There exists a function $f(x)$ belonging to Lip 1 and a constant C_4 such that

$$| H_n(f, 0) - f(0) | \geqslant C_4 \frac{\log n}{n}; \quad n = 2, 4, 6, \ldots .$$

In our proof of Theorem 1, we shall need the following lemmas. We shall henceforth write x_k for x_{kn} for the sake of simplicity.

Lemma 1 — Let x_j denote the nearest root to x, then

$$\left| \frac{(1 - x^2) \, U_{n-2}(x)}{(n - 1)(x - x_k)} \right| \leqslant \begin{cases} \dfrac{2}{i} & \text{if } k \neq j, \; k = j \pm i \\ 2 & \text{if } k = j \text{ or } j - 1 \end{cases}$$

PROOF : $$\left| \frac{(1 - x^2) \, U_{n-2}(x)}{(n - 1)(x - x_k)} \right| \leqslant \frac{2}{(n - 1)} \left| \frac{\sin n \left(\dfrac{\theta - \theta_k}{2} \right)}{\sin \left(\dfrac{\theta - \theta_k}{2} \right)} \right|$$

$$= \frac{2}{i} \quad \text{if } k \neq j, \; k = j \pm i$$

$$= 2 \quad \text{if } k = j \text{ or } j - 1$$

Lemma 2 — $$\sum_{k=2}^{n-1} \frac{1}{1 - x_k^2} = \sum_{k=2}^{n-1} \frac{1}{1 - x_k} = \sum_{k=2}^{n-1} \frac{1}{1 + x_k} = \frac{n(n - 2)}{3}$$

For the proof of Lemma 2 refer to Saxena (1974). We shall also use the following property of modulus of continuity, viz.

$$\omega(f, \lambda\delta) \leqslant (\lambda + 1)\, \omega(f, \delta); \ \lambda \geqslant 0. \qquad \ldots(1.4)$$

Proof of Theorem 1 — Now making suitable arrangements in the representation (1.2), after using Lemma 2, and then owing to the uniqueness of the polynomial, we obtain

$$\begin{aligned} H_n(f, x) - f(x) &= [f(1) - f(x)]\,(2 - x)\left[\frac{(1 + x)\, U_{n-2}(x)}{2(n-1)}\right]^2 \\ &\quad + [f(-1) - f(x)]\,(2 + x)\left[\frac{(1 - x)\, U_{n-2}(x)}{2(n-1)}\right]^2 \\ &\quad + \left[\frac{(1 + x)(1 - x^2)\, U^2_{n-2}(x)}{2(n-1)^2}\right] \sum_{k=2}^{n-1} \left[\frac{f(1) - f(x_k)}{1 - x_k}\right] \\ &\quad + \frac{(1 - x)(1 - x^2)\, U^2_{n-2}(x)}{2(n-1)^2} \sum_{k=2}^{n-1} \left[\frac{f(-1) - f(x_k)}{1 + x_k}\right] \\ &\quad + \sum_{k=2}^{n-1} [f(x_k) - f(x)]\left[\frac{(1 - x^2)\, U_{n-2}(x)}{(n-1)(x - x_k)}\right]^2 \\ &\quad + \sum_{k=1}^{n-1} \left[\frac{f(x_k) - f(x)}{x - x_k}\right] \cdot \left[\frac{x(1 - x^2)\, U^2_{n-2}(x)}{(n-1)^2}\right] \\ &= \sum_{k=1}^{6} S_k(x). \end{aligned} \qquad \ldots(2.1)$$

First we shall estimate $S_5(x)$. Thus, using Lemma 1, Property (1.4) and the fact that $|U_{n-2}(x)| \leqslant n - 1$, $|(1 - x^2)^{1/2}\, U_{n-2}(x)| \leqslant 1$, we have for $n > 1$

$$\begin{aligned} |S_5(x)| &\leqslant \sum_{\substack{k=j\pm i \\ k\neq j,\, j-1}} \omega\left[\frac{i(1 - x^2)^{1/2}\, |U_{n-2}(x)|}{n-1}\right] \\ &\quad \times \left[1 + \frac{(n-1)\,|x - x_k|}{i(1 - x^2)^{1/2}\, |U_{n-1}(x)|}\right]\left[\frac{(1 - x^2)\, U_{n-2}(x)}{(n-1)(x - x_k)}\right]^2 \\ &\quad + \omega\left[\frac{(1 - x^2)^{1/2}\, |U_{n-2}(x)|}{(n-1)}\right]\left[1 + \frac{(n-1)\,|(x_i - x)|}{(1 - x^2)^{1/2}\, |U_{n-2}(x)|}\right] \times \end{aligned}$$

(*equation continued on p.* 1332)

$$\times \left[\frac{(1-x^2)\,U_{n-2}(x)}{(n-1)(x-x_j)}\right]^2 + \omega\left[\frac{(1-x^2)^{1/2}\,|\,U_{n-2}(x)\,|}{n-1}\right]$$

$$\times \left[1 + \frac{(n-1)(x_{j-1}-x)}{(1-x^2)^{1/2}\,|\,U_{n-2}(x)\,|}\right]\left[\frac{(1-x^2)\,U_{n-2}(x)}{(n-1)(x-x_{j-1})}\right]^2$$

$$\leqslant 6 \sum_{i=1}^{n} \frac{1}{i^2}\,\omega\left[\frac{i(1-x^2)^{1/2}\,|\,U_{n-2}(x)\,|}{n-1}\right]. \qquad \ldots(2.2)$$

Similarly,

$$|\,S_6(x)\,| < 3\omega\left[\frac{(1-x^2)^{1/2}\,|\,U_{n-2}(x)\,|}{(n-1)}\right]. \qquad \ldots(2.3)$$

Again using Lemma 2 and proceeding as in estimate (2.2), we have

$$|\,S_k(x)\,| \leqslant \tfrac{4}{3}\,\omega\left[\frac{(1-x^2)^{1/2}\,|\,U_{n-2}(x)\,|}{(n-1)}\right];\ \text{if } k = 3, 4. \qquad \ldots(2.4)$$

For $k = 1, 2$ it is easy to show

$$|\,S_k(x)\,| \leqslant \tfrac{9}{4}\,\omega\left[\frac{(1-x^2)^{1/2}\,|\,U_{n-2}(x)\,|}{(n-1)^2}\right]. \qquad \ldots(2.5)$$

Thus taking into consideration (2.1) and estimates (2.2) - (2.5) we get,

$$|\,H_n(f, x) - f(x)\,| < 17 \sum_{i=1}^{n} \frac{1}{i^2}\,\omega\left[\frac{i(1-x^2)^{1/2}\,|\,U_{n-2}(x)\,|}{n-1}\right].$$

This completes the proof of Theorem 1.

Proof of Theorem 2 — Let $f(x) = |\,x\,|$ and $n = 2p - 2$ $(p \geqslant 2)$; then from (2.1) we have

$$H_n(f, 0) - f(0) = \frac{U^2_{n-2}(0)}{(n-1)^2} + 2\sum_{k=2}^{p-1} \frac{1}{(1+x_k)}\,\frac{U^2_{n-2}(0)}{(n-1)^2}$$

$$+ \frac{2U^2_{n-2}(0)}{(n-1)^2} \sum_{k=1}^{p-1} \frac{1}{x_k}$$

$$\geqslant \frac{2}{(n-1)^2} \sum_{k=1}^{p-1} \frac{1}{x_k}$$

and the theorem follows by a similar argument as in Vertesi (1971).

Remark : From Theorem 1 we can easily deduce the following three forms also:

(a) $$| H_n(f, x) - f(x) | = O(1)\, \omega\left[\frac{(1 - x^2)^{1/2}\, | U_{n-2}(x) | \log n}{n}\right]$$

(b) $$| H_n(f, x) - f(x) | = O(1) \sum_{i=1}^{n} \frac{1}{n}\, \omega\left[\frac{(1 - x^2)^{1/2}\, | U_{n-2}(x) |}{i}\right]$$

(c) $$| H_n(f, x) - f(x) | = O(1)\, \omega\left[\frac{(1 - x^2)^{1/2}\, | U_{n-2}(x) |}{n^{1-\epsilon}}\right]$$

$\epsilon > 0$, ϵ being small and fixed.

These forms are comparable with the corresponding forms given by Moldvan (1954), Bojanic (1972) and Telyakovskii (1966) or Gopengauz (1967).

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