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SUMS OF RECIPROCALS OF SOME MULTIPLICATIVE FUNCTIONS—II

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Let σ^* and φ^* denote the unitary analogues of the well-known σ and φ functions. In this paper, we establish a general theorem from which we deduce an asymptotic formula for the sum $\sum_{m \leqslant x} \frac{1}{\sigma^*(m)}$. We also establish an asymptotic formula for the sum $\sum_{m \leqslant x} \frac{1}{\varphi^*(m)}$

1. INTRODUCTION

Throughout this paper, m denotes a positive integer variable, p (with or without suffixes) denotes a prime and x denotes a real variable $\geqslant 3$. A divisor d of m is called unitary (Cohen 1960) if $(d, m/d) = 1$. We write $d \| m$ to mean that d is a unitary divisor of m.

Recently the authors (Ramaiah and Suryanarayana 1979) established an asymptotic formula for $\sum_{m \leqslant x} \frac{1}{\sigma(m)}$, where $\sigma(m)$ is the sum of the divisors of m and the asymptotic formula for $\sum_{m \leqslant x} \frac{1}{\varphi(m)}$, where φ is the Euler-totient function, was established by Landau (1900). In this paper, we establish the asymptotic formulae for the sums $\sum_{m \leqslant x} \frac{1}{\sigma^*(m)}$ and $\sum_{m \leqslant x} \frac{1}{\varphi^*(m)}$ where $\sigma^*(m)$ is the sum of the unitary divisors of m and $\varphi^*(m)$ is the unitary analogue (Cohen 1960) of the Euler-totient function, defined to be the number of positive integers $t \leqslant m$, satisfying $(t, m)^* = 1$, where the symbol $(t, m)^*$ denotes the greatest divisor of t which is also a unitary divisor of m. In fact, the asymptotic expression for $\sum_{m \leqslant x} \frac{1}{\sigma^*(m)}$ will be deduced as a corollary from a general result (see §3, Theorem 3.1).

For asymptotic formulae relating to the sums of reciprocals of certain arithmetical functions we refer to the papers of de Koninck (1972) and de Koninck and Galambos (1974).

2. Preliminaries

In this section, we state some known results and prove some lemmas which are required in the present discussion. Let $[x]$ denote, as usual, the largest integer $\leqslant x$. We need the following result due to Walfisz (1963).

Lemma 2.1 [cf. Walfisz 1963, (36), p. 144] — $\sum_{m\leqslant x} \frac{\mu(m)}{m}\rho\left(\frac{x}{m}\right) = O(\lambda(x))$,

where μ is the Möbius function,

$$\rho(x) = x - [x] - \tfrac{1}{2} \qquad \ldots(2.1)$$

and

$$\lambda(x) = \begin{cases} \log^{2/3} x\,(\log\log x)^{4/3}, & \text{if } x \geqslant 3 \\ 1, & \text{if } 0 < x < 3. \end{cases} \qquad \ldots(2.2)$$

Remark 2.1 : It is clear that $\lambda(x)$ is increasing for $x \geqslant 3$. Using this, it can be shown that if $x > 0$, then

$$\lambda(x) \leqslant H\lambda(y), \text{ for all } y \geqslant x$$

where H is an absolute positive constant.

Lemma 2.2 — For each fixed positive integer n,

$$F_n(x) \equiv \sum_{\substack{m\leqslant x \\ (m,n)=1}} \frac{\mu(m)}{m}\rho\left(\frac{x}{m}\right) = O(\sigma^*_{-1+\epsilon}(n)\,\lambda(x)) \qquad \ldots(2.3)$$

uniformly, for every $\epsilon > 0$, where (m, n) is the greatest common divisor of m and n, and $\sigma^*_t(n)$ is the sum of the t-th powers of the square-free divisors of n.

Proof: The proof of this lemma is quite similar to that of Lemma 3.5 of Suryanarayana and Prasad (1973). We have

$$F_n(x) = \sum_{m\leqslant x} \frac{\mu(m)}{m}\rho\left(\frac{x}{m}\right) \sum_{\substack{d\delta=m \\ d\mid n}} \mu(d)$$

$$= \sum_{\substack{d\delta\leqslant x \\ d\mid n \\ (d,\delta)=1}} \frac{\mu^2(d)\,\mu(\delta)\,\rho\left(\frac{x}{d\delta}\right)}{d\delta}$$

$$= \sum_{d\mid n} \frac{\mu^2(d)}{d} \sum_{\substack{\delta\leqslant x/d \\ (\delta,d)=1}} \frac{\mu(\delta)}{\delta}\rho\left(\frac{x}{d\delta}\right)$$

so that

$$F_n(x) = \sum_{d \mid n} \frac{\mu^2(d)}{d} F_d\left(\frac{x}{d}\right). \qquad \ldots(2.4)$$

If p is any prime with $(p, n) = 1$, we have

$$F_{pn}(x) = \sum_{\substack{m \leqslant x \\ (m, pn)=1}} \frac{\mu(m)}{m} \rho\left(\frac{x}{m}\right)$$

$$= \sum_{\substack{m \leqslant x \\ (m, n)=1}} \frac{\mu(m)}{m} \rho\left(\frac{x}{m}\right) - \sum_{\substack{m \leqslant x \\ (m, n)=1 \\ p \mid m}} \frac{\mu(m)}{m} \rho\left(\frac{x}{m}\right)$$

$$= F_n(x) - \sum_{\substack{pt \leqslant x \\ (t, n)=(t, p)=1}} \frac{\mu(p)\,\mu(t)\,\rho\left(\frac{x}{pt}\right)}{pt}$$

$$= F_n(x) + \frac{1}{p} \sum_{\substack{t \leqslant x \\ (t, np)=1}} \frac{\mu(t)}{t} \rho\left(\frac{x}{pt}\right)$$

$$= F_n(x) + \frac{1}{p} F_{pn}\left(\frac{x}{p}\right)$$

so that

$$F_{pn}(x) = F_n(x) + \frac{1}{p} F_{pn}\left(\frac{x}{p}\right).$$

Substituting $\frac{x}{p}, \frac{x}{p^2}, \ldots$, successively in the above we get

$$F_{pn}(x) = \sum_{i=0}^{c} \frac{1}{p^i} F_n\left(\frac{x}{p^i}\right) \qquad \ldots(2.5)$$

where $c = \left[\frac{\log x}{\log p}\right]$, since for $i \geqslant c + 1$,

$F_n\left(\frac{x}{p}\right) =$ an empty sum $= 0$. In particular, taking $n = 1$ in (2.5), we get

$$F_p(x) = \sum_{i=0}^{c} \frac{1}{p^i} F_1\left(\frac{x}{p^i}\right).$$

Hence by Lemma 2.1 and Remark 2.1, we have

$$F_p(x) = O\Big(H\lambda(x) \sum_{i=0}^{c} \frac{1}{p^i}\Big)$$

$$= O\Big(H\lambda(x) \sum_{i=0}^{\infty} \frac{1}{p^i}\Big) = O\Big(H\lambda(x) \frac{p}{p-1}\Big). \qquad \ldots(2.6)$$

Again, if p_1 and p_2 are distinct primes, then by (2.5),

$$F_{p_1p_2}(x) = \sum_{i=0}^{c_1} \frac{1}{p_1^i} F_{p_2}\Big(\frac{x}{p_1^i}\Big)$$

where $$c_1 = \Big[\frac{\log x}{\log p_1}\Big].$$

Now, by (2.6) and Remark 2.1, we have

$$F_{p_1p_2}(x) = O\Big(H \sum_{i=0}^{c_1} \frac{1}{p_1^i}\, \lambda\Big(\frac{x}{p_1^i}\Big) \frac{p_2}{p_2-1}\Big)$$

$$= O\Big(H^2\lambda(x) \frac{p_2}{p_2-1} \sum_{i=0}^{c_1} \frac{1}{p_1^i}\Big)$$

$$= O\Big(H^2\lambda(x) \frac{p_2}{p_2-1} \sum_{i=0}^{\infty} \frac{1}{p_1^i}\Big)$$

$$= O\Big(H^2\lambda(x) \frac{p_1}{p_1-1} \cdot \frac{p_2}{p_2-1}\Big).$$

Similarly, if $p_1, p_2, \ldots, p_r$ are distinct primes, then

$$F_{p_1p_2\cdots p_r}(x) = O\Big(\prod_{j=1}^{r} \frac{p_j}{p_j-1}\, \lambda(x)\, H^r\Big).$$

Thus for square-free d, we have

$$F_d(x) = O\Big(\frac{dH^{\omega(d)}}{\varphi(d)} \cdot \lambda(x)\Big) \qquad \ldots(2.7)$$

where $\omega(d)$ is the number of distinct prime factors of d. Now, by (2.4) and (2.7), it follows that

$$F_n(x) = O\left(\lambda(x) \sum_{d \mid n} \frac{\mu^2(d)\, H^{\omega(d)}}{\varphi(d)}\right). \qquad \ldots(2.8)$$

It is known that (cf. Hardy and Wright 1960, Theorem 327) $\frac{m^{1-(\epsilon/2)}}{\varphi(m)} \to 0$ as $m \to \infty$, for every $\epsilon > 0$. From this and the fact that $H^{\omega(d)} = O(d^{\epsilon/2})$ for every $\epsilon > 0$, it follows that

$$\sum_{d \mid n} \frac{\mu^2(d)\, d^{1-(\epsilon/2)} H^{\omega(d)}}{d^{1-(\epsilon/2)}\varphi(d)} = O\left(\sum_{d \mid n} \mu^2(d)\, H^{\omega(d)} d^{-1+(\epsilon/2)}\right)$$

$$= O\left(\sum_{d \mid n} \mu^2(d)\, d^{-1+\epsilon}\right) = O\left(\sigma^*_{-1+\epsilon}(n)\right).$$

Hence by (2.8), Lemma 2.2 follows.

Lemma 2.3 (cf. Ramaiah and Suryanarayana 1979, Lemma 2.2) — Let f be any multiplicative function satisfying

$$f(m) = O(m^{\epsilon}), \text{ for every } \epsilon > 0 \qquad \ldots(2.9)$$

and

$$f(p) + 1 = O(p^{-1/2}), \text{ for all primes } p. \qquad \ldots(2.10)$$

Further let h be the arithmetic function defined by

$$h(m) = \sum_{d \mid m} f(d). \qquad \ldots(2.11)$$

Then the series $\sum_{m=1}^{\infty} h(m)\, m^{-s}$ converges absolutely for any $s > 1/2$.

Lemma 2.4 — Under the hypothesis of Lemma 2.3, we have

$$\sum_{\substack{m \leqslant x \\ (m,n)=1}} \frac{f(m)}{m} = O(1),$$

where the O-constant is uniform in x and n.

PROOF: It is known that (cf. Davenport 1937; p. 10) $\sum_{\substack{m \leqslant x \\ (m,n)=1}} \mu(m)/m = O(1)$, where the O-constant is uniform in x and n. Also, by (2.11) and the Möbius inversion formula (cf. Hardy and Wright 1960; Theorem 266), we have

$$f(m) = \sum_{d \mid m} \mu(d)\, h(m/d). \qquad \ldots(2.12)$$

Now, by (2.12) and Lemma 2.3 we have

$$\sum_{\substack{m \leqslant x \\ (m,n)=1}} \frac{f(m)}{m} = \sum_{\substack{d\delta \leqslant x \\ (d,n)=(\delta,n)=1}} \frac{\mu(d)\, h(\delta)}{d\delta} = \sum_{\substack{\delta \leqslant x \\ (\delta,n)=1}} \frac{h(\delta)}{\delta} \sum_{\substack{d \leqslant x/\delta \\ (d,n)=1}} \frac{\mu(d)}{d}$$

$$= O\Big(\sum_{\delta < x} \frac{|\, h(\delta)\,|}{\delta} \Big) = O(1).$$

Hence Lemma 2.4 follows.

Lemma 2.5 — Under the hypothesis of Lemma 2.3, we have

$$G_n(x) \equiv \sum_{\substack{m \leqslant x \\ (m,n)=1}} \frac{f(m)}{m} \rho\left(\frac{x}{m}\right) = O(\sigma^*_{-1+\epsilon}(n)\, \lambda(x)) \qquad \ldots(2.13)$$

for every $\epsilon > 0$, where $\rho(x)$ and $\lambda(x)$ are given by (2.1) and (2.2) respectively; the O-constant being independent of x and n.

PROOF : By (2.12), Lemma 2.2, Remark 2.1 and Lemma 2.3, we have

$$G_n(x) = \sum_{\substack{d\delta \leqslant x \\ (d,n)=(\delta,n)=1}} \frac{\mu(d)\, h(\delta)}{d\delta} \rho\left(\frac{x}{d\delta}\right)$$

$$= \sum_{\substack{\delta \leqslant x \\ (\delta,n)=1}} \frac{h(\delta)}{\delta} \sum_{\substack{d \leqslant x/\delta \\ (d,n)=1}} \frac{\mu(d)}{d} \rho\left(\frac{x}{d\delta}\right)$$

$$= O\Big(\sigma^*_{-1+\epsilon}(n) \sum_{\delta \leqslant x} \frac{|\, h(\delta)\,|}{\delta} \lambda(x/\delta) \Big)$$

$$= O\Big(\sigma^*_{-1+\epsilon}(n)\, \lambda(x) \sum_{\delta \leqslant x} \frac{|\, h(\delta)\,|}{\delta} \Big)$$

$$= O(\sigma^*_{-1+\epsilon}(n)\, \lambda(x)).$$

Hence Lemma 2.5 follows.

Lemma 2.6 — Suppose g is a multiplicative function satisfying

$$g(p) = \frac{p}{p+1}, \text{ for all primes } p \qquad \ldots(2.14)$$

and

$$p^i \mid g(p^i) - 1 \mid \leqslant 1 \qquad \ldots(2.15)$$

for all prime powers p^j. Further, let g^* be the arithmetical function defined by

$$g^*(m) = \sum_{d \| m} \mu^*(d)\, g(m/d) \qquad \text{...(2.16)}$$

where $\quad \mu^*(m) = (-1)^{\omega(m)}$.

Then we have

$$\sum_{\substack{m \leqslant x \\ (m, n)=1}} g^*(m) = O(1) \qquad \text{...(2.17)}$$

where the O-constant is independent of x and n.

PROOF : From (2.16) it follows that for $j \geqslant 1$,

$$g^*(p^j) = g(p^j) - 1. \qquad \text{...(2.18)}$$

Since g is multiplicative, it follows that g^* is multiplicative (cf. Cohen 1960, Lemma 6.1). Hence by (2.18) and (2.15), we see that

$$m \mid g^*(m) \mid \leqslant 1, \text{ for all } m. \qquad \text{...(2.19)}$$

Further, by (2.18) and (2.14), we have

$$pg^*(p) = p(g(p) - 1) = p\left(\frac{p}{p+1} - 1\right) = -\frac{p}{p+1},$$

so that

$$1 + pg^*(p) = 1 - \frac{p}{p+1} = \frac{1}{p+1} = O(p^{-1}) = O(p^{-1/2}). \qquad \text{...(2.20)}$$

Hence if we take $f(m) = mg^*(m)$, from (2.19) and (2.20), it is clear that the conditions (2.9) and (2.10) are satisfied. Now, Lemma 2.6 follows from Lemma 2.4.

Lemma 2.7 — We have

$$\sum_{\substack{m \leqslant x \\ (m, n)=1}} g^*(m)\, \rho\left(\frac{x}{m}\right) = O(\sigma^*_{-1+\epsilon}(n)\, \lambda(x))$$

for every $\epsilon > 0$, where g^* is given by (2.16) and the O-constant is independent of x and n.

PROOF : Taking $f(m) = mg^*(m)$ in Lemma 2.5, we obtain Lemma 2.7 in virtue of (2.19) and (2.20).

Lemma 2.8 — Suppose n is square-free. Then we have

$$G^*_n(x) = \sum_{m \leqslant x} g^*(mn)\, \rho\left(\frac{x}{mn}\right) = O\left(\frac{\sigma^*_{-1+\epsilon}(n)\, \lambda(x)}{\varphi(n)}\right)$$

for every $\epsilon > 0$, where the O-constant is independent of x and n.

PROOF : Let $n = p_1 \dots p_k$. Then we have

$$G_n^*(x) = \sum_{\substack{0 \leqslant \alpha_i \leqslant [\log x/\log p_i] \\ 1 \leqslant i \leqslant k \\ p_1^{\alpha_1} \dots p_k^{\alpha_k} q \leqslant x \\ (q,n)=1}} g^*(p_1^{\alpha_1+1} \dots p_k^{\alpha_k+1} q)\, \rho\left(\frac{x}{p_1^{\alpha_1+1} \dots p_k^{\alpha_k+1} q}\right)$$

$$= \sum_{\substack{0 \leqslant \alpha_i \leqslant [\log x/\log p_i] \\ 1 \leqslant i \leqslant k}} g^*(p_1^{\alpha_1+1} \dots p_k^{\alpha_k+1}) \times \sum_{\substack{q \leqslant x/p_1^{\alpha_1} \dots p_k^{\alpha_k} \\ (q,n)=1}} g^*(q)\, \rho\left(\frac{x}{p_1^{\alpha_1+1} \dots p_k^{\alpha_k+1} q}\right) \qquad \dots(2.21)$$

Let $t_1 = x/p_1^{\alpha_1} \dots p_k^{\alpha_k}$ and $t_2 = x/p_1^{\alpha_1+1} \dots p_k^{\alpha_k+1}$. Then we have

$$\sum_{\substack{q \leqslant x/p_1^{\alpha_1} \dots p_k^{\alpha_k} \\ (q,n)=1}} g^*(q)\, \rho\left(\frac{x}{p_1^{\alpha_1+1} \dots p_k^{\alpha_k+1} q}\right) = \sum_{\substack{q \leqslant t_1 \\ (q,n)=1}} g^*(q)\, \rho\left(\frac{t_2}{q}\right)$$

$$= \sum_{\substack{q \leqslant t_2 \\ (q,n)=1}} g^*(q)\, \rho\left(\frac{t_2}{q}\right) + \sum_{\substack{t_2 < q \leqslant t_1 \\ (q,n)=1}} g^*(q)\, \rho\left(\frac{t_2}{q}\right)$$

$$= S_1 + S_2, \text{ say.} \qquad \dots(2.22)$$

By Lemma 2.7, we have

$$S_1 = O(\sigma^*_{-1+\epsilon}(n)\, \lambda(t_2)) = O(\sigma^*_{-1+\epsilon}(n)\, \lambda(x)). \qquad \dots(2.23)$$

For $q > t_2$, it follows from (2.1) that $\rho\left(\frac{t_2}{q}\right) = \frac{t_2}{q} - \frac{1}{2}$. Hence we have

$$S_2 = t_2 \sum_{\substack{t_1 \geqslant q > t_2 \\ (q,n)=1}} \frac{g^*(q)}{q} - \frac{1}{2} \sum_{\substack{t_2 < q < t_1 \\ (q,n)=1}} g^*(q). \qquad \dots(2.24)$$

Further by (2.19), we have

$$\sum_{\substack{q > t_2 \\ (q,n)=1}} \frac{g^*(q)}{q} = O\left(\sum_{q > t_2} \frac{|g^*(q)|}{q}\right)$$

$$= O\left(\sum_{q>t_2} \frac{1}{q^2}\right) = O\left(\frac{1}{t_2}\right)$$

so that by (2.17) and (2.24),

$$S_2 = O(1). \qquad \ldots(2.25)$$

Now, from (2.23), (2.25) and (2.22), we have

$$\sum_{\substack{q \leqslant x/p_1^{\alpha_1} \ldots p_k^{\alpha_k} \\ (q, n)=1}} g^*(q)\, \rho\left(\frac{x}{p_1^{\alpha_1+1} \ldots p_r^{\alpha_k+1}\, q}\right) = O(\sigma^*_{-1+\epsilon}(n)\, \lambda(x)).$$

Substituting this in (2.21), it follows from (2.19) that

$$G_n^*(x) = O\left(\sigma^*_{-1+\epsilon}(n)\, \lambda(x) \sum_{\substack{\alpha_i=0 \\ 1 \leqslant i \leqslant k}}^{\infty} \frac{1}{p_1^{\alpha_1+1} \ldots p_k^{\alpha_k+1}}\right)$$

$$= O\left(\sigma^*_{-1+\epsilon}(n)\, \lambda(x) \prod_{i=1}^{k} \frac{1}{p^i - 1}\right)$$

$$= O\left(\frac{\sigma^*_{-1+\epsilon}(n)\, \lambda(x)}{\varphi(n)}\right).$$

Hence Lemma 2.8 follows.

Lemma 2.9 (cf. Hardy and Wright 1960, Theorem 261) — We have

$$\phi(x; n) \equiv \sum_{\substack{m \leqslant x \\ (m, n)=1}} 1 = \sum_{d \mid n} \mu(d) \left[\frac{x}{d}\right].$$

Lemma 2.10 — We have

$$\sum_{m \leqslant x} \frac{g^*(m)\, \varphi(m)}{m^2} = A + O(x^{-1}),$$

where

$$A = \sum_{m=1}^{\infty} \frac{g^*(m)\, \varphi(m)}{m^2}. \qquad \ldots(2.26)$$

PROOF : The series $\sum_{m=1}^{\infty} \frac{g^*(m)\,\varphi(m)}{m^2}$ converges absolutely in virtue of (2.19) and the fact that $\varphi(m) \leqslant m$. For the same reasons we have

$$\sum_{m>x} \frac{|\,g^*(m)\,|\,\varphi(m)}{m^2} \leqslant \sum_{m>x} \frac{1}{m^2} = O(x^{-1}).$$

Hence Lemma 2.10 follows.

Now, we are in a position to prove the following:

Lemma 2.11 — Let g be given as in Lemma 2.6. Then we have

$$\sum_{m\leqslant x} g(m) = Ax + O(\lambda(x) \log x),$$

where A is given by (2.26).

PROOF : It follows from (2.16) and the unitary analogue of the Möbius inversion formula (cf. Cohen 1960, Theorem 2.3) that

$$g(m) = \sum_{d\,\|\,m} g^*(d). \qquad \ldots(2.27)$$

Now, by (2.27), Lemma 2.9 and (2.1), we have

$$\begin{aligned} \sum_{m\leqslant x} g(m) &= \sum_{\substack{d\delta\leqslant x \\ (d,\delta)=1}} g^*(d) = \sum_{d\leqslant x} g^*(d) \sum_{\substack{\delta\leqslant x/d \\ (\delta,d)=1}} 1 \\ &= \sum_{nt\leqslant x} g^*(nt)\, \mu(n) \left[\frac{x}{n^2 t}\right] \\ &= \sum_{nt\leqslant x} g^*(nt)\, \mu(n) \left\{\frac{x}{n^2 t} - \rho\left(\frac{x}{n^2 t}\right) - \tfrac{1}{2}\right\} \\ &= x \sum_{nt\leqslant x} \frac{g^*(nt)\,\mu(n)}{n^2 t} - \sum_{nt\leqslant x} \mu(n)\, g^*(nt)\, \rho\left(\frac{x}{n^2 t}\right) \\ &\qquad - \tfrac{1}{2} \sum_{nt\leqslant x} g^*(nt)\, \mu(n) \\ &= xS_3 - S_4 - \tfrac{1}{2} S_5, \text{ say.} \end{aligned} \qquad \ldots(2.28)$$

Now, by Lemma 2.10, we have

$$\begin{aligned} S_3 = \sum_{m\leqslant x} \frac{g^*(m)}{m} \sum_{nt=m} \frac{\mu(n)}{n} &= \sum_{m\leqslant x} \frac{g^*(m)\,\varphi(m)}{m^2} \\ &= A + O(x^{-1}). \end{aligned} \qquad \ldots(2.29)$$

Also, by Lemma 2.8, we have

$$S_4 = \sum_{n\leqslant x} \mu(n) \sum_{t\leqslant x/n} g^*(nt)\, \rho\left(\frac{x}{n^2t}\right)$$

$$= O\left(\lambda(x) \sum_{n\leqslant x} \frac{\sigma^*_{-1+\epsilon}(n)}{\varphi(n)}\right). \qquad \ldots(2.30)$$

Since $\dfrac{n}{\varphi(n)} = \sum\limits_{d\mid n} \dfrac{\mu^2(d)}{\varphi(d)}$, we have

$$\sum_{n\leqslant x} \frac{\sigma^*_{-1+\epsilon}(n)}{\varphi(n)} = \sum_{d\delta\leqslant x} \frac{\sigma^*_{-1+\epsilon}(d\delta)\,\mu^2 d}{d\delta\varphi(d)}$$

$$\leqslant \sum_{d\delta\leqslant x} \frac{\sigma^*_{-1+\epsilon}(d)\,\sigma^*_{-1+\epsilon}(\delta)}{d\delta\varphi(d)}$$

$$= \sum_{d\leqslant x} \frac{\sigma^*_{-1+\epsilon}(d)}{d\varphi(d)} \sum_{\delta\leqslant x/d} \frac{\sigma^*_{-1+\epsilon}(\delta)}{\delta}$$

$$= O\left(\log x \sum_{d\leqslant x} \frac{\sigma^*_{-1+\epsilon}(d)}{d\varphi(d)}\right)$$

where we used that $\sum\limits_{m\leqslant x} \dfrac{\sigma^*_{-1+\epsilon}(m)}{m} = O(\log x)$, which can be established using standard arguments. Since for every $\epsilon > 0$, $d^{1-\epsilon}/\varphi(d) \to 0$ as $d \to \infty$, and $\sigma^*_{-1+\epsilon}(d) \leqslant \tau(d)$, where $\tau(d)$ is the number of divisors of d, it follows that $\sum\limits_{d\leqslant x} \dfrac{\sigma^*_{-1+\epsilon}(d)}{d\varphi(d)} = O(1)$. Hence we have

$$\sum_{n\leqslant x} \frac{\sigma^*_{-1+\epsilon}(n)}{\varphi(n)} = O(\log x).$$

Substituting this in (2.30), we get that

$$S_4 = O(\lambda(x) \log x). \qquad \ldots(2.31)$$

Further, we have

$$S_5 = \sum_{nt \leqslant x} g^*(nt)\, \mu(n) = \sum_{m \leqslant x} g^*(m) \sum_{nt=m} \mu(n) = 1. \qquad \ldots(2.32)$$

Now, it follows from (2.28), (2.29), (2.31) and (2.32), that

$$\sum_{m \leqslant x} g(m) = Ax + O(1) + O(\lambda(x) \log x) + O(1)$$

$$= Ax + O(\lambda(x) \log x).$$

Hence Lemma 2.11 follows.

Lemma 2.12 [cf. Cohen 1960, Corollary 2.2.1 and cf. Cohen and Robinson 1963, eqn. (2.1)] — We have $\varphi^*(m)$ is multiplicative and

$$\varphi^*(m) = m \prod_{p^\alpha \parallel m} \left(1 - \frac{1}{p^\alpha}\right).$$

Lemma 2.13 (cf. Cohen 1961, Theorem 3.3, $m = n = 1$) — We have

$$\frac{1}{\varphi^*(m)} = m^{-1} \sum_{\substack{d\delta = m \\ (d, \delta)=1}} \frac{1}{\varphi^*(d)}.$$

Lemma 2.14 (cf. Suryanarayana 1970, Lemma 2.1) —We have

$$\sum_{\substack{m \leqslant x \\ (m, n)=1}} \frac{1}{m} = \frac{\varphi(n)}{n} (\log x + \gamma + \alpha(n)) + O\left(\frac{\theta(n)}{x}\right),$$

uniformly, where $\theta(n)$ is the number of square-free divisors of n,

$$\alpha(n) \equiv -\frac{n}{\varphi(n)} \sum_{d \mid n} \frac{\mu(d) \log d}{d} = \sum_{p \mid n} \frac{\log p}{p-1} \qquad \ldots(2.33)$$

and γ is Euler's constant.

Lemma 2.15 — For $s > 1$, we have

$$F(s) = \sum_{\substack{m=1 \\ (m, n)=1}}^{\infty} \frac{\varphi(m)}{m^s \varphi^*(m)} = \frac{a(s)}{A_s(n)} \qquad \ldots(2.34)$$

where

$$a(s) = \prod_p \left\{1 + \frac{(p-1)}{p} \sum_{j=1}^{\infty} \frac{1}{p^{j(s-1)}(p^j - 1)}\right\}$$

and

$$A_s(n) = \prod_{p \mid n} \left\{1 + \frac{(p-1)}{p} \sum_{j=1}^{\infty} \frac{1}{p^{j(s-1)}(p^j - 1)}\right\}.$$

PROOF : It follows from the definitions of the functions $\varphi(m)$ and $\varphi^*(m)$ that $\varphi^*(m) \geqslant \varphi(m)$. Hence the series $F(s)$ in (2.34) converges for $s > 1$. Since $\varphi(m)$ and $\varphi^*(m)$ are multiplicative, it follows that the general term of the series in (2.34) is multiplicative in m. Now, Lemma 2.15 follows on expanding the series $F(s)$ as an infinite product of Euler-type (cf. Hardy and Wright 1960, Theorem 286).

Lemma 2.16 — With the notation of lemma 2.15, we have for $s > 1$,

$$\sum_{m=1}^{\infty} \frac{\varphi(m) \log m}{m^s \varphi^*(m)} = a(s) \sum_{p} \frac{(p-1)}{p} \left\{1 + \frac{(p-1)}{p} \sum_{j=1}^{\infty} \frac{1}{p^{j(s-1)}(p^j - 1)}\right\}^{-1}$$
$$\times \left\{\sum_{j=1}^{\infty} \frac{j}{p^{j(s-1)}(p^j - 1)}\right\} \log p. \qquad \ldots(2.35)$$

PROOF : The series in (2.35) is uniformly convergent for $s \geqslant 1 + \epsilon > 1$. Now, the lemma follows from the termwise differentiation of the series in Lemma 2.15 $(n = 1)$ with respect to s.

In particular, taking $s = 2$ in Lemmas 2.15 and 2.16, we obtain the following:

$$\sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \frac{\varphi(m)}{m^2 \varphi^*(m)} = a/A(n) \qquad \ldots(2.36)$$

where

$$a = \prod_{p} \left\{1 + \frac{(p-1)}{p} \sum_{j=1}^{\infty} \frac{1}{p^j(p^j - 1)}\right\} \qquad \ldots(2.37)$$

and

$$A(n) = \prod_{p \mid n} \left\{1 + \frac{(p-1)}{p} \sum_{j=1}^{\infty} \frac{1}{p^j(p^j - 1)}\right\}. \qquad \ldots(2.38)$$

$$\sum_{m=1}^{\infty} \frac{\varphi(m) \log m}{m^2 \varphi^*(m)} = aB \qquad \ldots(2.39)$$

where

$$B = \sum_{p} \frac{(p-1)}{p} \left\{1 + \frac{(p-1)}{p} \sum_{j=1}^{\infty} \frac{1}{p^j(p^j - 1)}\right\}^{-1}$$

$$\times \left\{ \sum_{j=1}^{\infty} \frac{j}{p^j(p^j-1)} \right\} \log p. \qquad \ldots(2.40)$$

Lemma 2.17 — We have

$$\sum_{\substack{m \leqslant x \\ (m, n)=1}} \frac{\varphi(m)}{m^2 \varphi^*(m)} = a/A(n) + O(x^{-1}),$$

where a and $A(n)$ are given by (2.37) and (2.38).

PROOF : We have since $\varphi^*(m) \geqslant \varphi(m)$,

$$\sum_{m>x} \frac{\varphi(m)}{m^2 \varphi^*(m)} \leqslant \sum_{m>x} \frac{1}{m^2} = O(x^{-1}).$$

Now, Lemma 2.17 follows from (2.36).

Lemma 2.18 — We have

$$\sum_{m \leqslant x} \frac{\varphi(m) \log m}{m^2 \varphi^*(m)} = aB + O(x^{-1} \log x)$$

where a and B are given by (2.37) and (2.40).

PROOF : We have

$$\sum_{m>x} \frac{\varphi(m) \log m}{m^2 \varphi^*(m)} \leqslant \sum_{m>x} \frac{\log m}{m^2} = O(x^{-1} \log x).$$

Now, Lemma 2.18 follows from (2.39).

Lemma 2.19 — We have

$$\sum\nolimits_1 \equiv \sum_{\substack{m \leqslant x \\ p \mid m}} \frac{\varphi(m)}{m^2 \varphi^*(mp)} = \frac{a(p-1)\, A^*(p)}{p} + O(x^{-1}p^{-1})$$

where a is given by (2.37), and

$$A^*(p) = \sum_{\alpha=1}^{\infty} \frac{1}{p^{\alpha}(p^{\alpha+1}-1)A(p)} \qquad \ldots(2.41)$$

$A(p)$ being given by (2.38) ($n = p$).

PROOF : By Lemma 2.17, we have

$$\sum\nolimits_1 = \sum_{\substack{p^{\alpha} \leqslant x \\ \alpha \geqslant 1}} \frac{\varphi(p^{\alpha})}{p^{2\alpha}\varphi^*(p^{\alpha+1})} \Bigg(\sum_{\substack{t \leqslant x/p^{\alpha} \\ (t,p)=1}} \frac{\varphi(t)}{t^2\varphi^*(t)} \Bigg)$$

$$= \frac{(p-1)}{p} \sum_{\substack{p^{\alpha} \leqslant x \\ \alpha \geqslant 1}} \frac{1}{p^{\alpha}(p^{\alpha+1}-1)} \left\{ \frac{a}{A(p)} + O\left(\frac{p^{\alpha}}{x}\right) \right\}$$

$$= \frac{a(p-1)}{p} \sum_{\substack{p^{\alpha} \leqslant x \\ \alpha \geqslant 1}} \frac{1}{A(p)\, p^{\alpha}(p^{\alpha+1}-1)} + O\Bigg(x^{-1} \sum_{\substack{p^{\alpha} \leqslant x \\ \alpha \geqslant 1}} \frac{1}{(p^{\alpha+1}-1)} \Bigg)$$

$$= \frac{a(p-1)}{p} \sum_{\substack{p^{\alpha} \leqslant x \\ \alpha \geqslant 1}} \frac{1}{A(p)\, p^{\alpha}(p^{\alpha+1}-1)} + O(x^{-1}p^{-1}) \qquad \ldots(2.42)$$

since
$$\sum_{\substack{p^{\alpha} \leqslant x \\ \alpha \geqslant 1}} \frac{1}{p^{\alpha+1}-1} \leqslant \frac{2}{p} \sum_{\substack{p^{\alpha} \leqslant x \\ \alpha \geqslant 1}} \frac{1}{p^{\alpha}} \leqslant \frac{2}{p} \sum_{\alpha=1}^{\infty} \frac{1}{p^{\alpha}} = \frac{2}{p(p-1)}$$
$$= O(p^{-1}).$$

Further since $A(p) \geqslant 1$,

$$\sum_{\substack{p^{\alpha} > x \\ \alpha \geqslant 1}} \frac{1}{p^{\alpha}(p^{\alpha+1}-1)\, A(p)} \leqslant \sum_{\substack{p^{\alpha} > x \\ \alpha \geqslant 1}} \frac{1}{p^{\alpha}(p^{\alpha+1}-1)}$$
$$\leqslant \frac{2}{p} \sum_{p^{\alpha} > x} \frac{1}{p^{2\alpha}} \leqslant \frac{2}{p} \sum_{m > x} \frac{1}{m^2}$$
$$= O(p^{-1}x^{-1}). \qquad \ldots(2.43)$$

Now, Lemma 2.19, follows from (2.43) and (2.42).

Lemma 2.20 — We have

$$\sum\nolimits_2 = \sum_{m < x} \frac{\alpha(m)\, \varphi(m)}{m^2 \varphi^*(m)}$$

(*equation continued on p.* 1349)

$$= a \left\{ \sum_{p} \frac{\log p}{p^2(p-1)\,A(p)} + \sum_{p} \frac{A^*(p)\log p}{p^2} \right\} + O(x^{-1}),$$

where $\alpha(m)$ is given by (2.33), a, $A(p)$ and $A^*(p)$ are given by (2.37), (2.38) and (2.41) respectively.

PROOF : By (2.33), we have

$$\sum\nolimits_2 = \sum_{pt \leqslant x} \frac{\varphi(pt)\log p}{(p-1)\,p^2t^2\,\varphi^*(pt)}$$

$$= \sum_{\substack{pt \leqslant x \\ (p,t)=1}} \frac{\varphi(pt)\log p}{(p-1)\,p^2t^2\varphi^*(pt)} + \sum_{\substack{pt \leqslant x \\ p \mid t}} \frac{\varphi(pt)\log p}{(p-1)\,p^2t^2\varphi^*(pt)}$$

$$= \Sigma_3 + \Sigma_4, \text{ say.} \qquad \ldots(2.44)$$

Now, by Lemma 2.17, we have

$$\sum\nolimits_3 = \sum_{p \leqslant x} \frac{\varphi(p)\log p}{(p-1)\,p^2\varphi^*(p)} \sum_{\substack{t \leqslant x/p \\ (t,p)=1}} \frac{\varphi(t)}{t^2\varphi^*(t)}$$

$$= \sum_{p \leqslant x} \frac{\log p}{p^2(p-1)} \left\{ \frac{a}{A(p)} + O\left(\frac{p}{x}\right) \right\}$$

$$= a \sum_{p \leqslant x} \frac{\log p}{A(p)\,p^2(p-1)} + O\left(x^{-1} \sum_{p \leqslant x} \frac{\log p}{p(p-1)}\right)$$

$$= a \sum_{p} \frac{\log p}{A(p)\,p^2(p-1)} + O(x^{-2}\log x) + O(x^{-1}),$$

since we have

$$\sum_{p > x} \frac{\log p}{A(p)\,p^2(p-1)} \leqslant 2 \sum_{p > x} \frac{\log p}{p^3} \leqslant 2 \sum_{m > x} \frac{\log m}{m^3} = O(x^{-2}\log x)$$

and $\sum_{p \leqslant x} \frac{\log p}{p(p-1)} = O(1)$. Hence we have

$$\sum\nolimits_3 = a \sum_{p} \frac{\log p}{A(p)\,p^2(p-1)} + O(x^{-1}). \qquad \ldots(2.45)$$

For $p \mid t$, we have $\varphi(pt) = p\varphi(t)$. Hence by Lemma 2.19, we have

$$\sum\nolimits_4 = \sum_{p \leqslant x} \frac{\log p}{p(p-1)} \sum_{\substack{t \leqslant x/p \\ p \mid t}} \frac{\varphi(t)}{t^2\varphi^*(pt)}$$

(*equation continued on p.* 1350)

$$= \sum_{p \leqslant x} \frac{\log p}{p(p-1)} \left\{ \frac{a(p-1) A^*(p)}{p} + O(x^{-1}) \right\}$$

$$= a \sum_{p \leqslant x} \frac{A^*(p) \log p}{p^2} + O\left(x^{-1} \sum_{p \leqslant x} \frac{\log p}{p(p-1)} \right)$$

$$= a \sum_{p} \frac{A^*(p) \log p}{p^2} + O\left(\sum_{p > x} \frac{A^*(p) \log p}{p^2} \right) + O(x^{-1}). \quad \ldots(2.46)$$

By (2.41), we have

$$A^*(p) \leqslant \sum_{\alpha=1}^{\infty} \frac{1}{p^\alpha (p^{\alpha+1} - 1)} \leqslant \frac{2}{p} \sum_{\alpha=1}^{\infty} \frac{1}{p^{2\alpha}} = \frac{2}{p(p^2-1)} \leqslant \frac{1}{p}.$$

Hence we have

$$\sum_{p > x} \frac{A^*(p) \log p}{p^2} \leqslant \sum_{p > x} \frac{\log p}{p^3} = O\left(\sum_{m > x} \frac{\log m}{m^3} \right) = O(x^{-2} \log x).$$

Hence we have by (2.46),

$$\sum\nolimits_4 = a \sum_{p} \frac{A^*(p) \log p}{p^2} + O(x^{-1}). \quad \ldots(2.47)$$

Now, Lemma 2.20 follows from (2.44), (2.45) and (2.47).

Lemma 2.21 — We have

$$\sum\nolimits_5 = \sum_{m \leqslant x} \frac{\tau(m)}{\varphi^*(m)} = O(\log^2 x)$$

where $\tau(m)$ is the number of divisors of m.

PROOF : By Lemma 2.13, we have

$$\sum\nolimits_5 \leqslant \sum_{d\delta \leqslant x} \frac{\tau(d)\, \tau(\delta)}{d\delta\varphi^*(d)} \leqslant \sum_{d\delta \leqslant x} \frac{\tau(d)\, \tau(\delta)}{d\delta\varphi(d)}$$

$$= \sum_{d \leqslant x} \frac{\tau(d)}{d\varphi(d)} \sum_{\delta \leqslant x/d} \frac{\tau(\delta)}{\delta}$$

$$= O\left(\log^2 x \sum_{d \leqslant x} \frac{\tau(d)}{d\varphi(d)} \right)$$

$$= O(\log^2 x).$$

Hence Lemma 2.21 follows.

Main Results

First we have the following:

Theorem 3.1 — Let g be as given in Lemma 2.6. Then for $x \geqslant 3$,

$$\sum_{m \leqslant x} \frac{g(m)}{m} = A \log x + D + O(x^{-1} \log^{5/3} x (\log \log x)^{4/3}) \qquad \ldots(3.1)$$

where A is given by (2.26) and

$$D = A\gamma - \sum_{m=1}^{\infty} \frac{g^*(m)\, \varphi(m) \log m}{m^2} + \sum_{m=1}^{\infty} \frac{g^*(m)\, \varphi(m)\, \alpha(m)}{m^2} \qquad \ldots(3.2)$$

g^* and $\alpha(m)$ are given by (2.16) and (2.33).

Proof : Let $G(x) = \sum_{m \leqslant x} g(m)$ and $\triangle(x) = G(x) - Ax$. Then by Lemma 2.11, we have $\triangle(x) = O(\lambda(x) \log x)$, so that by partial summation, we get

$$\sum_{m \leqslant x} \frac{g(m)}{m} = A \log x + C + O(x^{-1}\lambda(x) \log x) \qquad \ldots(3.3)$$

where $C = A + \int_1^{\infty} \frac{\triangle(t)}{t^2}\, dt$, is a constant.

On the other hand, we have by (2.27) and Lemma 2.14

$$\sum_{m \leqslant x} \frac{g(m)}{m} = \sum_{\substack{d\delta \leqslant x \\ (d,\delta)=1}} \frac{g^*(d)}{d\delta} = \sum_{d \leqslant x} \frac{g^*(d)}{d} \sum_{\substack{\delta \leqslant x/d \\ (\delta, d)=1}} \frac{1}{\delta}$$

$$= \sum_{d \leqslant x} \frac{g^*(d)}{d} \left\{ \frac{\varphi(d)}{d} (\log x - \log d + \gamma + \alpha(d) + O(d\theta(d) x^{-1}) \right\}$$

$$= \sum_{d \leqslant x} \frac{g^*(d)\, \varphi(d)}{d^2} (\log x + \gamma)$$

$$- \sum_{d \leqslant x} \frac{g^*(d)\, \varphi(d) \log d}{d^2} + \sum_{d \leqslant x} \frac{g^*(d)\, \varphi(d)\, \alpha(d)}{d^2}$$

$$+ O(x^{-1} \sum_{d \leqslant x} | g^*(d) |\, \tau(d)).$$

By (2.19), we have

$$\sum_{d\leqslant x} | g^*(d) | \tau(d) \leqslant \sum_{d\leqslant x} \frac{\tau(d)}{d} = O(\log^2 x). \qquad \ldots(3.4)$$

Again by (2.19), we have

$$\sum_{d>x} \frac{| g^*(d) | \varphi(d) \log d}{d^2} \leqslant \sum_{d>x} \frac{\log d}{d^2} = O(x^{-1} \log x),$$

and hence

$$\sum_{d\leqslant x} \frac{g^*(d) \varphi(d) \log d}{d^2} = \sum_{d=1}^{\infty} \frac{g^*(d) \varphi(d) \log d}{d^2} + O(x^{-1} \log x). \qquad \ldots(3.5)$$

By the definition of $\alpha(m)$ given in (2.33), it can be easily shown that

$$\frac{\alpha(m) \varphi(m)}{m} = O(\tau(m)).$$

From this and by (2.19), we have

$$\sum_{d>x} \frac{| g^*(d) | \varphi(d) \alpha(d)}{d^2} = O\left(\sum_{d>x} \frac{\tau(d)}{d^2}\right) = O(x^{-1} \log x).$$

Hence we have

$$\sum_{d\leqslant x} \frac{g^*(d) \varphi(d) \alpha(d)}{d^2} = \sum_{d=1}^{\infty} \frac{g^*(d) \varphi(d) \alpha(d)}{d^2} + O(x^{-1} \log x). \qquad \ldots(3.6)$$

Now, by Lemma 2.10 and (3.4) – (3.6), we have

$$\begin{aligned}\sum_{m\leqslant x} \frac{g(m)}{m} &= A \log x + A\gamma - \sum_{d=1}^{\infty} \frac{g^*(d) \varphi(d) \log d}{d^2} \\ &\quad + \sum_{d=1}^{\infty} \frac{g^*(d) \varphi(d) \alpha(d)}{d^2} + O(x^{-1} \log^2 x) \\ &= A \log x + D + O(x^{-1} \log^2 x), \qquad \ldots(3.7)\end{aligned}$$

by (3.2). Now, comparing (3.3) and (3.7), we find that $C = D$, from which we obtain Theorem 3.1.

Corollary — For $x \geqslant 3$, we have

$$\sum_{m\leqslant x} \frac{1}{\sigma^*(m)} = B^* \log x + D^* + O(x^{-1} \log^{5/3} x (\log \log x)^{4/3}) \qquad \ldots(3.8)$$

where

$$B^* = \prod_p \left\{1 - \frac{p}{p-1} \sum_{j=1}^{\infty} \frac{1}{p^j(p^j+1)}\right\} \qquad \text{...(3.9)}$$

$$D^* = B^* \{\gamma + A_1 - A_2\} \qquad \text{...(3.10)}$$

where

$$A_1 = \sum_p \frac{pC^*(p) \log p}{(p-1)} \left\{\sum_{j=1}^{\infty} \frac{j}{p^j(p^{j+1}+1)}\right\} \qquad \text{...(3.11)}$$

$$A_2 = \sum_p \frac{C^*(p) \log p}{p^2} \left\{\sum_{j=0}^{\infty} \frac{1}{p^j(p^{j+1}+1)}\right\} \qquad \text{...(3.12)}$$

and

$$C^*(p) = \left\{1 - \frac{p}{p-1} \sum_{j=1}^{\infty} \frac{1}{p^j(p^{j+1}+1)}\right\}. \qquad \text{...(3.13)}$$

PROOF: We take $g(m) = m/\sigma^*(m)$. Since $\sigma^*(p^j) = p^j + 1$, we have for $j \geqslant 1$

$$g^*(p^j) = g(p^j) - 1 = \frac{p^j}{p^j+1} - 1 = -\frac{1}{p^j+1} \qquad \text{...(3.14)}$$

so that

$p^j \mid g(p^j) - 1 \mid \leqslant 1$. Hence g satisfies (2.15) and the condition (2.14) is clearly satisfied. Making use of (3.14), it is not difficult to show that

$$A = \sum_{m=1}^{\infty} \frac{g^*(m)\, \varphi(m)}{m^2} = B^* \qquad \text{...(3.15)}$$

$$\sum_{m=1}^{\infty} \frac{g^*(m)\, \varphi(m) \log m}{m^2} = -B^* A_1 \qquad \text{...(3.16)}$$

and

$$\sum_{m=1}^{\infty} \frac{g^*(m)\, \varphi(m)\, \alpha(m)}{m^2} = -B^* A_2 \qquad \text{...(3.17)}$$

where B^*, A_1 and A_2 are given by (3.9), (3.11) and (3.12) respectively. Now, we obtain (3.8) by taking $g(m) = m/\sigma^*(m)$ in (3.1) and from (3.15) – (3.17).

Theorem 3.2 — We have

$$\sum_{m\leqslant x} \frac{1}{\varphi^*(m)} = a \log x + E + O(x^{-1} \log^2 x) \qquad \ldots(3.18)$$

where

$$E = a\left\{\gamma - B + \sum_{p} \frac{\log p}{p^2(p-1)\,A(p)} + \sum_{p} \frac{A^*(p) \log p}{p^2}\right\} \qquad \ldots(3.19)$$

where a, $A(p)$, B and $A^*(p)$ are respectively given by (2.37), (2.38), (2.40) and (2.41).

PROOF : By Lemmas 2.13 and 2.14, we have

$$\sum_{m\leqslant x} \frac{1}{\varphi^*(m)} = \sum_{\substack{d\delta\leqslant x\\(d,\delta)=1}} \frac{1}{d\delta\varphi^*(d)} = \sum_{d\leqslant x} \frac{1}{d\varphi^*(d)} \sum_{\substack{\delta\leqslant x/d\\(\delta,d)=1}} \frac{1}{\delta}$$

$$= \sum_{d\leqslant x} \frac{1}{d\varphi^*(d)} \left\{\frac{\varphi(d)}{d}(\log x - \log d + \gamma + \alpha(d) + O(x^{-1} d\theta(d))\right\}$$

$$= (\log x + \gamma) \sum_{d\leqslant x} \frac{\varphi(d)}{d^2\varphi^*(d)} - \sum_{d\leqslant x} \frac{\varphi(d) \log d}{d^2\varphi^*(d)}$$

$$+ \sum_{d\leqslant x} \frac{\varphi(d)\,\alpha(d)}{d^2\varphi^*(d)} + O\left(x^{-1} \sum_{d\leqslant x} \frac{\theta(d)}{\varphi^*(d)}\right).$$

Now, by Lemmas 2.17 ($n = 1$), 2.18, 2.20 and (3.19) we have

$$\sum_{m\leqslant x} \frac{1}{\varphi^*(m)} = a \log x + E + O(x^{-1} \log x) + O\left(x^{-1} \sum_{d\leqslant x} \frac{\theta(d)}{\varphi^*(d)}\right).$$

Now, since $\sum_{d\leqslant x} \frac{\theta(d)}{\varphi^*(d)} \leqslant \sum_{d\leqslant x} \frac{\tau(d)}{\varphi^*(d)} = O(\log^2 x)$, by Lemma 2.21, we obtain (3.18).

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