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NEW CRITERIA FOR p-VALENCE

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In this paper we consider the classes $T_{n+p-1}(\alpha)$ of functions

$$f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \ldots,$$

regular in the unit disc E and satisfying

$$\text{Re}\,\frac{(D^{n+p-1}f)'}{pz^{p-1}} > \alpha,\ 0 \leqslant \alpha < 1,\ z \in E$$

where $D^{n+p-1}f(z) = \dfrac{z^p}{(1-z)^{n+p}} * f(z)$. It is proved that

$$T_{n+p}(\alpha) \subset T_{n+p-1}(\alpha).$$

Since $T_0(\alpha)$ is the class of functions $f(z)$, with $\text{Re}\,\dfrac{f'(z)}{pz^{p-1}} > \alpha$ all functions in $T_{n+p-1}(\alpha)$ are p-valent.

1. INTRODUCTION

Let $A(p)$ denote the class of functions

$$f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \ldots, \qquad \ldots(1.1)$$

p a positive integer which are regular in the unit disc $E = \{z : |z| < 1\}$. Let

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}z^{p+n}$$

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n}z^{p+n}$$

belong to $A(p)$. We denote the Hadamard product or the convolution of f and g by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}b_{p+n}z^{p+n}.$$

In this paper we shall prove that a function $f \in A(p)$ and satisfy one of the conditions

$$\text{Re}\,\frac{(D^{n+p-1}f(z))'}{pz^{p-1}} > \alpha,\ 0 \leqslant \alpha < 1,\ z \in E \qquad \ldots(1.2)$$

n any integer greater than p, where ( )′ stands for the first derivative and

$$D^{n+p-1}f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z) \qquad ...(1.3)$$

is p-valent in E. We denote by $T_{n+p-1}(\alpha)$ the classes of functions $f(z) \in A(p)$ and satisfying (1.2). It is easy to see that

$$z(D^{n+p-1}f(z))' = (n+p)D^{n+p}f(z) - nD^{n+p-1}f(z). \qquad ...(1.4)$$

Using (1.4) condition (1.2) can be re-written in the form

$$\text{Re}\left\{(n+p)\frac{D^{n+p}f(z)}{pz^p} - n\frac{D^{n+p-1}f(z)}{pz^p}\right\} > \alpha, \quad z \in E. \qquad ...(1.5)$$

We shall show that

$$T_{n+p}(\alpha) \subset T_{n+p-1}(\alpha). \qquad ...(1.6)$$

Since $T_0(\alpha)$ is the class of functions which satisfy the condition

$$\text{Re}\,\frac{f'(z)}{pz^{p-1}} > \alpha \geqslant 0$$

and we know from (Umezawa 1957) that such functions are p-valent, the p-valence of functions in $T_{n+p-1}(\alpha)$ follows from (1.6).

By putting $p = 1$, we shall get the criteria for univalence.

2. The Classes $T_{n+p-1}(\alpha)$

Theorem 1 — $T_{n+p}(\alpha) \subset T_{n+p-1}(\alpha)$, $0 \leqslant \alpha < 1$, n is any integer greater than p.

We need the following lemma due to Jack (1971).

Lemma 1 — Let $w(z)$ be non-constant and regular in $|z| < 1$, $w(0) = 0$. Then if $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$, at z_0, we can write

$$z_0 w'(z_0) = k w(z_0)$$

where k is a real number greater than or equal to one.

Proof of Theorem 1 — Let $f(z) \in T_{n+p}(\alpha)$. Define a regular function $w(z)$ in E such that $w(0) = 0$, $w(z) \neq -1$ by

$$(n+p)\,D^{n+p}f(z) - nD^{n+p-1}f(z) = pz^p\,\frac{1+(2\alpha-1)\,w(z)}{1+w(z)}. \qquad ...(2.1)$$

Differentiating (2.1), we get

$$\frac{(D^{n+p}f(z))'}{pz^{p-1}} = \frac{1+(2\alpha-1)\,w(z)}{1+w(z)} - \frac{2(1-\alpha)}{n+p}\,\frac{zw'(z)}{(1+w(z))^2} \qquad ...(2.2)$$

We claim that $|w(z)| < 1$ for all $z \in E$. For, otherwise, by Lemma 1, there exists a z_0, $|z_0| < 1$ such that

$$z_0 w'(z_0) = kw(z_0) \qquad ...(2.3)$$

with $|w(z_0)| = 1$ and $k \geqslant 1$.

(2.2) in conjunction with (2.3) gives

$$\frac{(D^{n+p}f(z_0))'}{pz_0^{p-1}} = \frac{1+(2\alpha-1)\,w(z_0)}{1+w(z_0)} - \frac{2(1-\alpha)}{n+p}\,\frac{kw(z_0)}{(1+w(z_0))^2}. \qquad ...(2.4)$$

Since $\mathrm{Re}\,\frac{1+(2\alpha-1)\,w(z_0)}{1+w(z_0)} = \alpha$, $k \geqslant 1$ and $\frac{w(z_0)}{(1+w(z_0))^2}$ is real and positive we see that $\mathrm{Re}\,\frac{(D^{n+p}f(z_0))'}{pz_0^{p-1}} < \alpha$. This contradicts the hypothesis that $f(z) \in T_{n+p}(\alpha)$.

Hence $|w(z)| < 1$, $z \in E$ and it follows from (2.1) that $f(z) \in T_{n+p-1}(\alpha)$.

Theorem 2 — Let c be any real number greater than p. If $f(z) \in T_{n+p-1}(\alpha)$, then

$$F(z) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)\,dt \in T_{n+p-1}(\alpha), \text{ for } c+p>0.$$

PROOF : One can easily verify that the function $F(z)$ satisfies

$$z(D^{n+p-1}F(z))' = (p+c)\,D^{n+p-1}f(z) - cD^{n+p-1}F(z). \qquad ...(2.5)$$

Define a regular function $w(z)$ in E by

$$\frac{(D^{n+p-1}F(z))'}{z^{p-1}} = p\,\frac{1+(2\alpha-1)\,w(z)}{1+w(z)}. \qquad ...(2.6)$$

Obviously $w(0) = 0$, $w(z) \neq -1$ for $z \in E$.

Using (1.4), (2.6) can be re-written as

$$(n+p)\,D^{n+p}F(z) - nD^{n+p-1}F(z) = pz^p\,\frac{1+(2\alpha-1)\,w(z)}{1+w(z)}. \qquad ...(2.7)$$

Differentiating (2.7) and using (2.5), we get

$$\frac{(D^{n+p-1}f(z))'}{pz^{p-1}} = \frac{1+(2\alpha-1)\,w(z)}{1+w(z)} - \frac{2(1-\alpha)}{p+c}\,\frac{zw'(z)}{(1+w(z))^2}. \qquad ...(2.8)$$

Now proceeding as in Theorem 1, we can show that $F(z) \in T_{n+p-1}(\alpha)$.

Theorem 3 — Let $f(z) \in A(p)$ and satisfy the condition

$$\mathrm{Re}\,\frac{(D^{n+p-1}f(z))'}{pz^{p-1}} > \alpha - \frac{(1-\alpha)}{2(p+c)},\; c+p>0.$$

Then the function

$$F(z) = \frac{p+c}{z^c} \int_0^z t^{c-1} f(t)\, dt \in T_{n+p-1}(\alpha).$$

The proof of this theorem is similar to that of Theorem 2 and so we omit it.

Corollary 3(*a*) — By putting $n + p = c = 1$ and $\alpha = 0$ in Theorem 3, it follows that if $f(z) \in A(p)$ and satisfies the condition

$$\operatorname{Re} \frac{f'(z)}{pz^{p-1}} > -\frac{1}{2(p+1)}$$

then

$$\operatorname{Re} \frac{F'(z)}{pz^{p-1}} > 0$$

and hence $F(z)$ is p-valent in E.

Corollary 3(*b*) — Taking $n + p = c = 1$ and $\alpha = 1/(2p + 3)$, we see that if

$$\operatorname{Re} \frac{f'(z)}{pz^{p-1}} > 0, \text{ then } \operatorname{Re} \frac{F'(z)}{pz^{p-1}} > \frac{1}{2p+3}.$$

Remark : By taking $p = 1$ in Corollary 3(*a*) and Corollary 3(*b*) we get the following results :

(i) $\operatorname{Re} f'(z) > -\frac{1}{4}$ implies $\operatorname{Re} F'(z) > 0$;

(ii) $\operatorname{Re} f'(z) > 0$ implies $\operatorname{Re} F'(z) > \frac{1}{5}$.

Both these results are extensions of an earlier result due to Libera (1965) viz.; $\operatorname{Re} f'(z) > 0$ implies $\operatorname{Re} F'(z) > 0$.

3. Converse of Theorem 2

In this section we prove the converse of Theorem 2.

Theorem 4 — Let $c + p > 0$ and $f(z)$ be defined by

$$F(z) = \frac{p+c}{z^c} \int_0^z t^{c-1} f(t)\, dt, \; c + p > 0.$$

If $F(z) \in T_{n+p-1}(\alpha)$, then $f(z) \in T_{n+p-1}(\alpha)$ in $|z| < \frac{p+c}{1 + \sqrt{(p+c)^2 + 1}}$. The result is sharp.

PROOF : Since $F(z) \in T_{n+p-1}(\alpha)$, we can write

$$z(D^{n+p-1}F(z)) = pz^p\, [\alpha + (1 - \alpha)\, u(z)] \qquad \ldots(3.1)$$

where $u(z) \in P$, the class of functions with positive real part in the unit disc E and normalized by $u(0) = 1$. We can re-write (3.1) as

$$(n + p) D^{n+p}F(z) - nD^{n+p-1}F(z) = pz^p [\alpha + (1 - \alpha) u(z)]. \quad ...(3.2)$$

Differentiating (3.2) and making use of (2.5) we get after a simple computation

$$\left(\frac{(D^{n+p-1}f(z))'}{pz^{p-1}} - \alpha\right)(1 - \alpha)^{-1} = u(z) + \frac{1}{p + c} zu'(z). \quad ...(3.3)$$

Using the well-known estimate $| zu'(z) | \leqslant \frac{2r}{1 - r^2} \operatorname{Re} u(z)$, $| z | = r$, (3.3) yields

$$\operatorname{Re}\left\{\left(\frac{(D^{n+p-1}f(z))'}{pz^{p-1}} - \alpha\right)(1 - \alpha)^{-1}\right\} \geqslant \left(1 - \frac{1}{p + c}\frac{2r}{1 - r^2}\right) \operatorname{Re} u(z). \quad ...(3.4)$$

The right-hand side of (3.4) is positive if $r < \frac{p + c}{1 + \sqrt{(p + c)^2 + 1}}$. The result is sharp for the function $f(z)$ defined by

$$f(z) = \frac{z^{1-c}}{p + c} (z^c F(z))'$$

where $F(z)$ is given by

$$(D^{n+p-1}F(z))' = pz^{p-1}\left(\frac{1 + (2\alpha - 1) z}{1 + z}\right).$$

Corollary 4(*a*) — By Putting $n + p = 1$ and $\alpha = 0$, we see that if

$$\operatorname{Re} \frac{F'(z)}{pz^{p-1}} > 0,\ z \in E, \text{ then } \operatorname{Re} \frac{f'(z)}{pz^{p-1}} > 0 \text{ for } | z | < \frac{p + c}{1 + \sqrt{(p + c)^2 + 1}}.$$

By putting $c = 1$, we get the result obtained by Goel (1972).

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