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BERNOULLI POLYNOMIALS OF THE SECOND KIND AND GENERAL ORDER

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In this paper, the authors study a polynomial set $b_n^{(\mu)}(x)$ which is a Sheffer set for the invertible shift invariant operator $\left(\frac{D}{\Delta}\right)^{\mu}$ and the basic sequence $(x)_n$. For $\mu = 1$, it reduces to the set of Bernoulli polynomials of the second kind studied in the usual manner by Jordan (1950) and Richardson (1954). The present treatment is based on a systematic use of the finite operator calculus rigorously developed by Rota *et al.* (1973).

1. INTRODUCTION

Bernoulli polynomials and Bernoulli numbers have been a subject of extensive investigations and have been sufficiently used in various branches of mathematics such as theory of numbers, analysis, calculus of finite differences, statistics, combinatorial analysis, etc. These polynomials have been extended and generalized in various directions; in particular, Bernoulli polynomials of general order have been discussed by Erdélyi (1953, 1955), Milne-Thomson (1951) and Luke (1969). Jordan (1950) and Richardson (1954) have extensively studied Bernoulli polynomials of the second kind.

The aim of this paper is to employ systematically the finite operator calculus developed by Rota *et al.* (1973) to study polynomials $b_n^{(\mu)}(x)$ which form a Sheffer set with respect to invertible shift invariant operator $\left(\frac{D}{\Delta}\right)^{\mu}$ and basic sequence $(x)_n$. These polynomials will be called Bernoulli polynomials of the second kind and order μ. For $\mu = 1$, these reduce to ordinary Bernoulli polynomials of second kind and some of the results proved for $b_n^{(\mu)}(x)$ reduce to known results of Jordan (1950) and Richardson (1954). However, several results obtained by putting $\mu = 1$ are believed to be new even for ordinary Bernoulli polynomials of the second kind.

In section 2 we prove results on $b_n^{(\mu)}(x)$ mainly using the fact that it is a Sheffer set. In section 3 we show that $b_n^{(\mu)}(x)$ is a cross-sequence and derive its properties from this consideration.

Throughout this paper we use the terminology and notation of Rota *et al.* (1973).

2. Polynomial $b_n^{(\mu)}(x)$ as a Sheffer Set

We define $b_n^{(\mu)}(x)$ to be the sheffer set corresponding to the basic sequence $(x)_n = x(x-1)\dots(x-n+1)$ and the invertible shift invariant operator

$$S^{\mu} = \left(\frac{D}{\triangle}\right)^{\mu} = \left(\frac{D}{e^{D}-1}\right)^{\mu} \qquad \text{...(2.1)}$$

where μ is any integer, D the ordinary differential operator d/dx and $\triangle$ the difference operator. Obviously the delta operator for this set is $\triangle$. Since $b_n^{(\mu)}(x)$ is a Sheffer set corresponding to S^{μ}, we have (Rota *et al.* 1973, §5, Prop. 1)

$$b_n^{(\mu)}(x) = S^{-\mu}(x)_n = \left(\frac{\triangle}{D}\right)^{\mu}(x)_n. \qquad \text{...(2.2)}$$

Moreover

$$\triangle\, b_n^{(\mu)}(x) = n b_{n-1}^{(\mu)}(x) \qquad \text{...(2.3)}$$

which for a positive integer k leads to

$$\triangle^{k} b_n^{(\mu)}(x) = (n)_k b_{n-k}^{(\mu)}(x). \qquad \text{...(2.4)}$$

Also,

$$D b_n^{(\mu)}(x) = n b_{n-1}^{(\mu-1)}(x) \qquad \text{...(2.5)}$$

and

$$D^{\mu} b_n^{(\mu)}(x) = (n)_{\mu}(x)_{n-\mu}. \qquad \text{...(2.6)}$$

The binomial theorem for the Sheffer polynomials [Rota *et al.* 1973, §5, Prop. 2] yields the identity

$$b_n^{(\mu)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} b_k^{(\mu)}(x)\,(y)_{n-k}. \qquad \text{...(2.7)}$$

Putting $y = 1$ in (2.7) we get

$$b_n^{(\mu)}(x+1) = n b_{n-1}^{(\mu)}(x) + b_n^{(\mu)}(x). \qquad \text{...(2.8)}$$

The substitution $y = 0$ after interchange of x and y in (2.7) gives

$$b_n^{(\mu)}(x) = \sum_{k=0}^{n} \binom{n}{k} b_k^{(\mu)} (x)_{n-k} \qquad \text{...(2.9)}$$

where $b_k^{(\mu)} = b_k^{(\mu)}(0)$, will be called the Bernoulli number of second kind and of order μ. This result is analogous to the well-known representation of Bernoulli polynomials $B_n(x)$ in terms of Bernoulli numbers B_n and also to its generalisation to order μ.

Theorem 1 — For $n \geqslant 2$,

$$(n - x)\, b_n^{(\mu)}(x) = \mu n b_{n-1}^{(\mu)}(x-1) + (\mu - x)\, b_n^{(\mu)}(x-1) - \mu b_n^{(\mu+1)}(x+1) \qquad \text{...(2.10)}$$

PROOF: We shall prove this result using Pincherle derivative [Rota *et al.* 1973, §4]. Now

$$\left[\left(\frac{\Delta}{D}\right)^{\mu}\right]' f(x) = \left[\left(\frac{\Delta}{D}\right)^{\mu} X - X\left(\frac{\Delta}{D}\right)^{\mu}\right] f(x)$$

or

$$\left(\frac{\Delta}{D}\right)^{\mu} x f(x) = \left\{\left[\left(\frac{\Delta}{D}\right)^{\mu}\right]' + x\left(\frac{\Delta}{D}\right)^{\mu}\right\} f(x)$$

$$= \left\{\mu\left[\left(\frac{\Delta}{D}\right)^{\mu} + \frac{1}{\Delta}\left(\frac{\Delta}{D}\right)^{\mu} - \frac{1}{\Delta}\left(\frac{\Delta}{D}\right)^{\mu+1}\right] + x\left(\frac{\Delta}{D}\right)^{\mu}\right\} f(x)$$

which on taking $f(x) = (x - 1)_{n-1}$ gives

$$b_n^{(\mu)}(x) = \mu b_{n-1}^{(\mu)}(x-1) + \frac{1}{\Delta} b_{n-1}^{(\mu)}(x-1) - \frac{1}{\Delta} b_{n-1}^{(\mu+1)}(x-1) + x b_{n-1}^{(\mu)}(x-1).$$

Operating both sides by the operator Δ and using (2.3), we obtain

$$n b_{n-1}^{(\mu)}(x) = \mu(n-1)\, b_{n-2}^{(\mu)}(x-1) + \mu b_n^{(\mu)}(x-1) - \mu b_{(n-1)}^{(\mu+1)}(x-1) + (E - I)\left[x b_n^{(\mu)}(x-1)\right]$$

$$= \mu(n-1)\, b_{n-2}^{(\mu)}(x-1) + \mu b_{n-1}^{(\mu)}(x-1) - \mu b_{n-1}^{(\mu+1)}(x-1) + (x+1)\, b_{n-1}^{(\mu)}(x) - x b_{n-1}^{(\mu)}(x-1)$$

or

$$(n - x - 1)\, b_{n-1}^{(\mu)}(x) = \mu(n-1)\, b_{n-2}^{(\mu)}(x-1) + (\mu - x)\, b_{n-1}^{(\mu)}(x-1)$$
$$- \mu b_{n-1}^{(\mu+1)}(x+1).$$

Replacing n by $n+1$, we get

$$(n - x)\, b_n^{(\mu)}(x) = \mu n b_{n-1}^{(\mu)}(x-1) + (\mu - x)\, b_n^{(\mu)}(x-1) - \mu b_n^{(\mu+1)}(x-1)$$

which is the required result.

Next, replacing x by $x+1$ in (2.10) and using (2.8) we get

$$(n - \mu)\, b_n^{(\mu)}(x) = n(\mu - n + x + 1)\, b_{n-1}^{(\mu)}(x) - \mu b_n^{(\mu+1)}(x). \qquad (2.11)$$

Putting $x = 0$ in it we get

$$b_n^{(\mu+1)} = \frac{1}{\mu}\,(n(\mu + 1 - n)\, b_{n-1}^{(\mu)} + (\mu - n)\, b_n^{\mu}. \qquad ...(2.12)$$

Generating Relation for $b_n^{(\mu)}(x)$

Since $b_n^{(\mu)}(x)$ is a Sheffer set, its generating relation is given by (Rota *et al.* 1973, §5, Prop. 5)

$$\sum_{n\geqslant 0} b_n^{(\mu)}(x)\,\frac{t^n}{n!} = \frac{1}{s(q^{-1}(t))}\; e^{xq^{-1}(t)} \qquad ...(2.13)$$

where $s(t) = \left(\frac{t}{e^t - t}\right)$, $q(t) = e^t - 1$ are the indicators of S^{μ} and $\triangle$ and $q^{-1}(t)$ is the formal power series inverse to $q(t)$.

Evidently $q^{-1}(t) = \log(1 + t)$ and hence we get the generating relation

$$\sum_{n\geqslant 0} b_n^{(\mu)}(x)\,\frac{t^n}{n!} = \left(\frac{t}{\log(1+t)}\right)^{\mu} (1 + t)^x. \qquad ...(2.14)$$

Putting $x = 0$ in (2.14), we get the generating relation for Bernoulli numbers of second kind and order μ

$$\sum_{n>0} b_n^{(\mu)}\,\frac{t^n}{n!} = \left(\frac{t}{\log(1+t)}\right)^{\mu}. \qquad ...(2.15)$$

Theorem 2 — If μ is a positive integer, then

$$b_n^{(\mu)}(\mu + x) = \sum_{k=0}^{\mu} \binom{\mu}{k} (n)_k \, b_{n-k}^{(\mu)}(x). \qquad \ldots(2.16)$$

PROOF : We will prove this result using the generating relation for $b_n^{(\mu)}(x)$. From (2.14), we have

$$\sum_{n \geqslant 0} b_n^{(\mu)}(\mu + x) \frac{t^n}{n!} = \left(\frac{t}{\log(1+t)}\right)^{\mu} (1+t)^{\mu+x}$$

$$= (1+t)^{\mu} \left(\frac{t}{\log(1+t)}\right)^{\mu} (1+t)^{x}$$

$$= (1+t)^{\mu} \sum_{n \geqslant 0} b_n^{(\mu)}(x) \frac{t^n}{n!}$$

$$= \sum_{n \geqslant 0} \sum_{k=0}^{\mu} \binom{\mu}{k} t^k \,.\, b_n^{(\mu)}(x) \frac{t^n}{n!}$$

$$= \sum_{n \geqslant 0} \sum_{k=0}^{\mu} \binom{\mu}{k} b_n^{(\mu)}(x) \frac{t^{n+k}}{n!}.$$

Equating the coefficients of t^n, we get the required result (2.16).

Symmetry of the Polynomial $b_n^{(\mu)}(x)$

Putting $x = n - 1 + y$ in the expression for $b_{2n-1}^{(\mu)}(x)$, we get

$$b_{2n-1}^{(\mu)}(n - 1 + y) = \left(\frac{\Delta}{D}\right)^{\mu} (n - 1 + y)_{2n-1}$$

$$= \left(\frac{\Delta}{D}\right)^{\mu} (n - 1 - y)_{2n-1}$$

$$= b_{2n-1}^{(\mu)}(n - 1 - y)$$

$$b_{2n-1}^{(\mu)}(x) = b_{2n-1}^{(\mu)}(2n - 2 - x). \qquad \ldots(2.17)$$

Hence, the polynomials $b_{2n-1}^{(\mu)}(x)$ of odd degree are symmetrical and the axis of symmetry is $x = n - 1$.

Now, replacing n by $n + 1$ in (2.17) we get

$$b_{2n+1}^{(\mu)}(x) = b_{2n+1}^{(\mu)}(2n - x).$$

Operating both sides of this equation by $\triangle$, we get

$$b_{2n}^{(\mu)}(x) = -b_{2n}^{(\mu)}(2n - x - 1). \qquad \ldots(2.18)$$

Replacing x by $n - \frac{1}{2} + x$ we get

$$b_{2n}^{(\mu)}(n - \tfrac{1}{2} + x) = -b_{2n}^{(\mu)}(n - \tfrac{1}{2} - x)$$

and this is the statement of symmetry for polynomials of even degree.

For $x = n - \frac{1}{2}$ we get from (2.18)

$$b_{2n}^{(\mu)}(n - \tfrac{1}{2}) = 0.$$

This shows that the polynomials of even degree are symmetrical with respect to the point $x = n - \frac{1}{2}$.

3. $b_n^{(\mu)}(x)$ AS A CROSS-SEQUENCE

If the parameter μ ranges over the field of scalars, then the shift invariant operators $S^{-\mu} = (\triangle/D)^{\mu}$ form a one parameter group so that

$$b_n^{[\mu]}(x) = b_n^{(\mu)}(x)$$

is a cross-sequence and

$$b_n^{[\mu]}(x) = S^{-\mu}(x)_n. \qquad \ldots(3.1)$$

Consequently [Rota *et al.* 1973, §8]

$$b_n^{[\mu+\nu]}(x + y) = \sum_{k=0}^{n} \binom{n}{k} b_k^{[\mu]}(x)\, b_{n-k}^{[\nu]}(y). \qquad \ldots(3.2)$$

for all μ and ν in the field and for all x and y.

For $\mu = 1, \nu = \mu - 1$, we get from (3.2)

$$b_n^{[\mu]}(x + y) = \sum_{k=0}^{n} \binom{n}{k} b_k(x)\, b_{n-k}^{[\mu-1]}(y). \qquad \ldots(3.3)$$

Putting $y = 0$ in (3.3) after interchanging x and y, we get

$$b_n^{[\mu]}(x) = \sum_{k=0}^{n} \binom{n}{k} b_k b_{n-k}^{[\mu-1]}(x). \qquad ...(3.4)$$

Substituting $\mu = -\nu$ in (3.2) we get

$$(x+y)_n = \sum_{k=0}^{n} \binom{n}{k} b_k^{[\mu]}(x)\, b_{n-k}^{[-\mu]}(y). \qquad ...(3.5)$$

Putting $y = 0$ in (3.5) after interchanging x and y we get

$$(x)_n = \sum_{k=0}^{n} \binom{n}{k} b_k^{[\mu]}\, b_{n-k}^{[-\mu]}(x). \qquad ...(3.6)$$

For $\mu = \nu$, (3.2) becomes

$$b_n^{[2\mu]}(x+y) = \sum_{k=0}^{n} \binom{n}{k} b_k^{[\mu]}(x)\, b_{n-k}^{[\mu]}(y). \qquad ...(3.7)$$

But $\quad b_n^{[2\mu]}(x+y) = S^{-2\mu}(x+y)_n$

$$= \left(\frac{\Delta}{D}\right)^{2\mu}(x+y)_n$$

which gives $D^{\mu} b_n^{[2\mu]}(x+y) = (n)_\mu\, b_{n-\mu}^{[\mu]}(x+y)$.

Substituting the value of $b_n^{[2\mu]}(x+y)$ in (3.7) and using (2.6), we get

$$(n)_\mu\, b_{n-\mu}^{[\mu]}(x+y) = \sum_{k=0}^{n} \binom{n}{k} (k)_\mu\, (x)_{k-\mu}\, b_{n-k}^{[\mu]}(y).$$

Replacing $n - \mu$ by n we get

$$b_n^{[\mu]}(x+y) = \frac{1}{(n+\mu)_\mu} \sum_{k=0}^{n+\mu} \binom{n+\mu}{k} (k)_\mu\, (x)_{k-\mu}\, b_{n+\mu-k}^{[\mu]}(y). \qquad ...(3.8)$$

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