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A GENERALIZED TWO-POINT BOUNDARY VALUE PROBLEM

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In this paper the authors arrive at a necessary and sufficient condition for the existence of a solution of the boundary value problem $y'' = f(t, y, y')$ satisfying most general boundary conditions.

INTRODUCTION

A study of two-point boundary value problems associated with second order differential equations plays an important role in the theory of differential equations and its application to physical problems. Historically, many theories and techniques developed for these problems arose naturally from initial value problems. It is well known that differential inequalities (Jackson 1968), fixed point theorems (Agarwal 1974), and Picard's iteration are the techniques that are commonly used for both initial and boundary value problems.

Barr and Miletta (1975) have given a necessary and sufficient condition for the uniqueness of solutions to a certain class of two-point boundary value problems. But the problems considered by them do not exhaust all possible boundary conditions. In this paper we arrive at a necessary and sufficient condition for the existence of a solution of the boundary value problem associated with a second order differential equation satisfying the most general boundary conditions since one encounters physical problems involving general boundary conditions quite often. By giving particular values to the boundary condition matrix, namely, $\alpha_{11} = 1$, $\alpha_{24} = 1$ and the remaining α_{ij}'s zero, considered in this paper, we obtain the results of Barr and Miletta (1975) as a special case.

2. A NECESSARY AND SUFFICIENT CONDITION

We consider the boundary value problem

$$y'' = f(t, y, y') \qquad ...(2.1)$$

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \end{pmatrix} \begin{pmatrix} y(a) \\ y(b) \\ y'(a) \\ y'(b) \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad ...(2.2)$$

where $f(t, y, y')$ is assumed to be continuous on $[a, b] \times R^2$, $a \leqslant t_1 < t_2 \leqslant b$, $-\infty < \alpha_{ij} < \infty$, and where solutions of the initial value problems associated with

(2.1) exist, are unique, and also assume that the solutions are defined throughout $[a, b]$. Let $\phi_1(t)$ and $\phi_2(t)$ be solutions of (2.1) satisfying (2.2). Then $\phi(t) = \phi_1(t) - \phi_2(t)$ satisfies the equation

$$y'' = F(t, y, y') \quad \ldots(2.3)$$

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \end{pmatrix} \begin{pmatrix} y(a) \\ y(b) \\ y'(a) \\ y'(b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \ldots(2.4)$$

where $F(t, \phi, \phi') = f(t, \phi(t) + \phi_2(t), \phi'(t) + \phi_2'(t)) - \phi_2''(t)$. Since $F(t, 0, 0) = 0$ we have $\phi(t) = 0$ is a solution of (2.3) and (2.4). Thus we have the following result.

Result 2.1 — The problem (2.3) satisfying (2.4) has a solution $\phi = 0$ if and only if $\phi + \phi_2$ is a solution of (2.1) and (2.2).

Theorem 2.1 — Let $C: \{x: [a, b] \times R^2 \to R / (x''(t) \leqslant M > 0 \;\forall\; t \in [a, b], \; | x'(a) | \leqslant M$ and $| x(a) | \leqslant M\}$. There exists a solution to (2.3) satisfying (2.4) iff

$$\inf_{x(t) \in C} \int_a^b | x''(t) - F(t, x(t), x'(t)) | \, dt = 0.$$

PROOF: If there exists a solution to (2.3) satisfying (2.4), then for some $M > 0$, the above infimum is obviously zero. Conversely, suppose the above infimum is zero. Let $\{x_k\}$, $k = 1, 2, \ldots$, be a sequence of functions in C such that

$$\lim_{k \to \infty} \int_a^b | x_k(t) - \int_a^t \int_a^\tau F(t, x_k(t), x_k'(t)) \, dt \, d\tau = 0.$$

The sequence of functions $\{x_k'(t)\}$, $k = 1, 2, \ldots$ can easily be seen to be equi-continuous and uniformly bounded, for any t_1 and t_2. We have

$$| x_k'(t_1) - x_k'(t_2) | = \left| [\int_a^{t_1} x_k''(t) - \int_a^{t_2} | x_k''(t)] \, dt \right|$$

$$\leqslant \int_{t_1}^{t_2} | x_k''(t) | \, dt$$

$$\leqslant M(t_2 - t_1)$$

$$| x_k'(t) | \leqslant \left| \int_a^t x_k''(t) \, dt \right| + M$$

$$\leqslant M(b - a) + M.$$

Similarly

$$| x_k(t_1) - x_k(t_2) | \leqslant \int_{t_1}^{t_2} | x'_k(t) | \, dt$$

$$\leqslant M(t_2 - t_1)^2 + M(t_2 - t_1)$$

$$\leqslant M(b - a)^2 + M(b - a)$$

and $$| x_k(t) | \leqslant M(b - a)^2 + M(b - a) + M.$$

Therefore there exist uniformly convergent subsequences $\{x'_k(t)\}$ and $\{x_k(t)\}$ which we can again denote without loss of generality by $\{x'_k(t)\}$ and $\{x_k(t)\}$ converging uniformly on $[a, b]$. Let us denote the limit function by $x(t)$. We have then

$$\alpha_{11}x_k(a) + \alpha_{12}x_k(b) + \alpha_{13}x'_k(a) + \alpha_{14}x'_k(b) = 0$$

$$\alpha_{21}x_k(a) + \alpha_{22}x_k(b) + \alpha_{23}x'_k(a) + \alpha_{24}x'_k(b) = 0$$

and $$x''_k(t) = F(t, x_k(t), x'_k(t)).$$

But x_k and x'_k converge uniformly to x and x', since F is continuous. It then follows that $x''(t) = F(t, x(t), x'(t))$,

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \end{pmatrix} \begin{pmatrix} x(a) \\ x(b) \\ x'(a) \\ x'(b) \end{pmatrix} = 0.$$

Thus x is a solution.

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References

Agarwal, R. P. (1974). Two-point problems for nonlinear third order differential equations. *J. Math. Phys. Sci.*, **8**, 571.

Bailey, P., Shampine, L., and Waltman, P. (1968). Nonlinear two-point boundary value problems. Mathematics in Science and Engineering, Vol. 44, Academic Press, New York. *MR* **37** # 6524.

Barr, D., and Miletta, P. (1975). A necessary and sufficient condition for uniqueness of solutions to two-point boundary value problems. *Pacific J. Math.*, **57**, 325–30.

Jackson, L. K. (1968). Subfunctions and second order ordinary differential inequalities. *Adv. Math.*, **2**, 307–63.