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FLOW PAST A POROUS SPHERICAL SHELL WITH VARIABLE PERMEABILITY USING MATCHED ASYMPTOTIC TECHNIQUE

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The flow of fluid past a porous spherical shell when permeability at any point of the shell varies as some power of its radial distance from the centre is investigated. Approximate solutions are obtained with the help of matched asymptotic technique. Drag on the shell and hence on the sphere is calculated. Several limiting and particular cases are derived.

FORMULATION OF THE PROBLEM

We consider a porous spherical shell with variable permeability of internal radius b and external radius a, immersed in a uniform stream of velocity U. Law of variation of permeability is taken to be $K = kr^m$, where $b \leqslant r \leqslant a$. We divide the flow field into three regions.

Region I : Inner most region which is the interior of the shell and full of viscous liquid with radius b,

Region II : Porous region of the shell where the Darcy's Law will hold good,

Region III : External region, which is outside the spherical shell.

Regions I and III are free fluid regions governed by Navier-Stokes equations. We further subdivide the external region III into (i) Stokes' region (near the surface of shell) and (ii) Oseen's region (away from the surface of shell).

We choose (r, θ, φ) as the spherical polar coordinates with the axis $\theta = 0$ chosen to be in the direction of the free stream velocity U. Let $\mathbf{q}$ and $\mathbf{Q}$ be the velocity vectors in the fluid and the porous medium respectively.

Introducing the non-dimensional quantitities

$$\bar{\mathbf{q}}_i = \frac{\mathbf{q}_i}{U}, \bar{r} = \frac{r}{a}, \bar{p}_i = \frac{ap_i}{\mu U}, \bar{P} = \frac{aP}{\mu U}, R = \frac{Ua}{\nu},$$

$$\bar{\mathbf{Q}} = \frac{\mathbf{Q}}{U}, \sigma = \frac{b}{a}, \bar{K} = \frac{K}{a^2} = \bar{K}_s\ (\bar{r})^m,$$

where $\bar{K}_s$ is non-dimensional permeability on the surface of the shell called non-dimensional surface permeability, the equations of motion in the free flow region and porous medium on dropping the bars are reduced to the following non-dimensional form

$$R(\mathbf{q}_i \cdot \vec{\nabla})\, \mathbf{q}_i + \vec{\nabla} p_i = \nabla^2 \mathbf{q}_i \qquad \ldots(1)$$

$$\vec{\nabla} \cdot \mathbf{q}_i = 0 \qquad \ldots(2)$$

$$Q_r = -K_s r^m \frac{\partial P}{\partial r} \qquad \ldots(3)$$

$$Q_\theta = -K_s r^{m-1} \frac{\partial P}{\partial \theta} \qquad \ldots(4)$$

and
$$\vec{\nabla} \cdot \mathbf{Q} = 0, \qquad \ldots(5)$$

where $i = 1$ and $i = z$ are taken respectively for the set of equations in region I and region III.

The boundary conditions in non-dimensional form at the interface of porous region and free fluid region are

(i) continuity of pressure, ...(6)

(ii) continuity of the normal velocity, ...(7)

(iii) $e_{r\theta} = \beta'(q_{\theta_2} - Q_\theta)$ at $r = 1$, ...(8)

(iv) $\tilde{e}_{r\theta} = -\beta''(q_{\theta_1} - Q_\theta)$ at $r = \sigma$ ...(9)

where

$$\beta' = \frac{\alpha}{\sqrt{K_s}} \text{ and } \beta'' = \frac{\beta'}{\sqrt{\sigma^m}},$$

$$e_{r\theta} = r\frac{\partial}{\partial r}\left(\frac{q_{\theta_2}}{r}\right) + \frac{1}{r}\frac{\partial q_{r_2}}{\partial \theta},\ \tilde{e}_{r\theta} = r\frac{\partial}{\partial r}\left(\frac{q_{\theta_1}}{r}\right) + \frac{1}{r}\frac{\partial q_{r_1}}{\partial \theta}$$

and α is a constant depending upon the porous material.

SOLUTION

We introduce non-dimensional Stokes stream function $\tilde{\Psi}$, $\bar{\Psi}$ and Ψ for the regions I, II and III respectively, thus the non-dimensional equations of motion in regions I, II and III in these stream functions are given by

$$\frac{1}{r^2}\frac{\partial(\tilde{\Psi}, D_r^2 \tilde{\Psi})}{\partial(r, c)} + \frac{2}{r^2} D_r^2 \tilde{\Psi} L_r \tilde{\Psi} = \frac{1}{R} D_r^4 \tilde{\Psi} \qquad \ldots(10)$$

$$\frac{\partial^2 \bar{\Psi}}{\partial \theta^2} - \cot\theta \frac{\partial \bar{\Psi}}{\partial \theta} - mr\frac{\partial \bar{\Psi}}{\partial r} + r^2 \frac{\partial^2 \bar{\Psi}}{\partial r^2} = 0 \qquad \ldots(11)$$

and

$$\frac{1}{r^2}\frac{\partial(\Psi, D_r^2 \Psi)}{\partial(r, c)} + \frac{2}{r^2} D_r^2 \Psi \, L_r \Psi = \frac{1}{R} D_r^4 \Psi \qquad \ldots(12)$$

where $c = \cos\theta \qquad \ldots(13)$

$$D_r^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1-c^2}{r^2}\frac{\partial^2}{\partial c^2} \qquad \ldots(14)$$

$$L_r = \frac{c}{1-c^2} \cdot \frac{\partial}{\partial r} + \frac{1}{r} \cdot \frac{\partial}{\partial c}. \qquad \ldots(15)$$

We assume the following expansions valid for small R

$$(\tilde{\Psi}, \bar{\Psi}, \Psi) = (\tilde{\Psi}_0, \bar{\Psi}_0, \Psi_0) + R(\tilde{\Psi}_1, \bar{\Psi}_1, \Psi_1) + R^2(\tilde{\Psi}_2, \bar{\Psi}_2, \Psi_2) + \ldots \qquad \ldots(16)$$

and

$$(p, p_1, p_2) = (p_0, p_{10}, p_{20}) + R(p_1, p_{11}, p_{21}) + \ldots \,. \qquad \ldots(17)$$

Expansion (16) satisfy eqns. (10) to (12) respectively and boundary conditions (6) to (9).

Equation (16) holds in the Stokes region where r is $O(1)$, we choose the Oseen variables as

$$\rho = Rr \text{ and } \Psi = R^2\Psi. \qquad \ldots(18)$$

Thus eqn. (12) in Oseen variable is reduced to

$$\frac{1}{\rho}\frac{\partial(\Psi, D_\rho^2 \Psi)}{\partial(\rho, c)} + \frac{2}{\rho^2} D_\rho^2 \Psi \, L_\rho \Psi = D_\rho^4 \Psi. \qquad \ldots(19)$$

The outer expansion which we call as Oseen expansion is taken to be

$$\Psi(R, \rho, c) = \Psi_0(\rho, c) + F_1(R)\, \Psi_1(\rho, c) + F_2(R)\, \Psi_2(\rho, c) + \ldots \qquad \ldots(20)$$

where $\dfrac{F_{n+1}(R)}{F_n(R)} \to 0 \text{ as } R \to 0. \qquad \ldots(21)$

The expansion (20) is required to satisfy the equation (19) and uniform stream condition at infinity. The slip condition on the surface of porous sphere is replaced by the condition that (20) should be matched to the Stokes expansion (16) for Ψ at small value of ρ.

Following the matching technique as given by Van Dyke (1972) and used by Verma and Bhatt (1975) the leading and second terms of the expansions (16) and (17) are determined as

$$\tilde{\Psi}_0 = (\bar{a}r^4 + \bar{b}r^2)(1 - c^2) \quad \text{...(22)}$$

$$\bar{\Psi}_0 = (Ar^g + (Br^h)(1 - c^2) \quad \text{...(23)}$$

$$\Psi_0 = \tfrac{1}{2}\left(r^2 - \frac{b_1}{r} - d_1 r\right)(1 - c^2) \quad \text{...(24)}$$

$$p_{10} = 20\,\bar{a}rc \quad \text{...(25)}$$

$$p_0 = -\frac{(Agr^{-h} + Bhr^{-g})\,c}{K_s} \quad \text{...(26)}$$

$$p_{20} = -\frac{d_1 c}{r^2} + (p_{20})_\infty \quad \text{...(27)}$$

$$\tilde{\Psi}_1 = \tfrac{1}{4} d_1 \tilde{\Psi}_0 + (M_3 r^3 + M_5 r^5)\, Q_2(c) \quad \text{...(28)}$$

$$\bar{\Psi}_1 = \tfrac{1}{4} d_1 \bar{\Psi}_0 + (\bar{A}r^G + \bar{B}r^H)\, Q_2(c) \quad \text{...(29)}$$

$$\Psi_1 = \frac{d_1}{4}\Psi_0 + \left\{L_1 + \frac{L_2}{r^2} + \frac{d_1}{4}\left(r^2 - d_1 r + \frac{b_1}{r}\right)\right\} Q_2(c) \quad \text{...(30)}$$

$$p_{11} = (1 - 3\cos^2\theta)\left[7M_5\frac{r^2}{2} + \frac{4}{3}\bar{a}^2 r^4 + 20\bar{a}\bar{b}r^2\right] + \frac{2}{3}\bar{a}^2 r^4 + 5\bar{a}d_1\cos\theta r + E_1 \quad \text{...(31)}$$

$$p_1 = \frac{-d_1}{4}\,\frac{(Agr^{-h} + Bhr^{-g})\cos\theta}{K_s} - \frac{1}{12K_s}(\bar{A}Gr^{-H} + \bar{B}Hr^{-G}) \times (1 - 3\cos^2\theta) + E_2 \quad \text{...(32)}$$

$$p_{21} = \cos^2\theta\left(\frac{-3L_1}{r^3} - \frac{3}{2}\,\frac{b_1 d_1}{r^4} + \frac{d_1^2}{2r^2} + \frac{3b_1}{2r^3} - \frac{3b_1^2}{8r^6}\right) + \left(\frac{-b_1}{2r^3} + \frac{L_1}{r^3} - \frac{b_1^2}{8r^6} - \frac{d_1^2}{4r^2} + \frac{b_1 d_1}{4r^4} - \frac{d_1^2}{4r^2}\cos\theta + (p_{21})_\infty \quad \text{...(33)}$$

where

$$Q_2(c) = \frac{c}{2}(c^2 - 1),\; g = \frac{(1 + m) + \sqrt{(1 + m)^2 + 8}}{2}$$

$$h = \frac{(1 + m) - \sqrt{(1 + m)^2 + 8}}{2},\; G = \frac{(1 + m) + \sqrt{(1 + m)^2 + 24}}{2}$$

$$H = \frac{(1 + m) - \sqrt{(1 + m)^2 + 24}}{2}.$$

Using the boundary conditions (6) to (9) the constants are

$$\bar{a} = \frac{3(2+\beta')}{20\sigma(\varphi_2\varphi_3 - \varphi_1\varphi_4)}[g\varphi_4\sigma^{-h} - h\varphi_3\sigma^{-g}], \quad ...(34)$$

$$\bar{b} = \frac{3(2+\beta')[20K_s\varphi_3\sigma^{h-2} - 20K_s\varphi_4\sigma^{g-2} - g\varphi_4\sigma^{1-h} + h\varphi_3\sigma^{1-g}]}{20(\varphi_2\varphi_3 - \varphi_1\varphi_4)} \quad ...(35)$$

$$b_1 = \frac{3(2+\beta')[(2K_s+g)\varphi_4 - (2K_s+h)\varphi_3]}{(\varphi_2\varphi_3 - \varphi_1\varphi_4)} + 1 \quad ...(36)$$

$$d_1 = \frac{3(2+\beta')}{\varphi_2\varphi_3 - \varphi_1\varphi_4}(\varphi_3 h - \varphi_4 g) \quad ...(37)$$

$$A = \frac{-3(2+\beta')K_s\varphi_4}{(\varphi_2\varphi_3 - \varphi_1\varphi_4)} \quad ...(38)$$

$$B = \frac{3(2+\beta')K_s\varphi_3}{(\varphi_2\varphi_3 - \varphi_1\varphi_4)} \quad ...(39)$$

where

$$\varphi_1 = 2\beta' g + 2K_s\beta' g + 6g + 12K_s + 2\beta' K_s \quad ...(40)$$

$$\varphi_2 = 2\beta' h + 2K_s\beta' h + 6h + 12K_s + 2\beta' K_s \quad ...(41)$$

$$\varphi_3 = \frac{3}{10K_s} g\sigma^{-h} - \frac{2\beta'}{\sqrt{\sigma^m}}(\sigma)^{g-2} + \frac{1}{10K_s}\frac{\beta'}{\sqrt{\sigma^m}} g(\sigma)^{1-h} + \frac{\beta'}{\sqrt{\sigma^m}} \cdot g(\sigma)^{g-2} \quad ...(42)$$

and

$$\varphi_4 = \frac{3}{10K_s} h\sigma^{-g} - \frac{2\beta'}{\sqrt{\sigma^m}}(\sigma)^{h-2} + \frac{1}{12K_s}\frac{\beta'}{\sqrt{\sigma^m}} h(\sigma)^{1-g} + \frac{\beta'}{\sqrt{\sigma^m}} h(\sigma)^{h-2}. \quad ...(43)$$

$$L_1 = \bar{A}\theta_1 + \bar{B}\theta_2 - \theta_3 \quad ...(44)$$

$$L_2 = \bar{A}(1-\theta_1) + \bar{B}(1-\theta_2) + \theta_3 - \theta_4 \quad ...(45)$$

$$M_3 = \bar{A}(\sigma^{G-3} - \sigma^2\theta_3) + \bar{B}(\sigma^{H-3} - \sigma^2\theta_6) \quad ...(46)$$

$$M_5 = \bar{A}\theta_5 + \bar{B}\theta_6 \quad ...(47)$$

$$E_1 = -\left[\frac{2}{3}\bar{a}^2\sigma^4 + \frac{2b_1^2}{8} + \frac{d_1^2}{12} + \frac{b_1 d_1}{4}\right] \quad ...(48)$$

$$E_2 = -\left(\frac{2b_1^2}{8} + \frac{d_1^2}{12} + \frac{b_1 d_1}{4}\right) \quad ...(49)$$

where

$$\bar{A} = \frac{(\theta_8 + 12\theta_3)(Z_1\sigma^{-G-2} + 42\theta_6) + \theta_7(Z_1 + 12\theta_2)}{(Z_2 + 12\theta_1)(Z_1\sigma^{-G-2} + 42\theta_6) - (Z_2\sigma^{-H-2} + 42\theta_5)(Z_1 + 12\theta_2)} \quad \ldots(50)$$

$$\bar{B} = \frac{(\theta_8 + 12\theta_3)(Z_2\sigma^{-H-2} + 42\theta_5) + \theta_7(Z_2 + 12\theta_1)}{(Z_1 + 12\theta_2)(Z_2\sigma^{-H-2} + 42\theta_5) - (Z_1\sigma^{-G-2} + 42\theta_6)(Z_2 + 12\theta_1)}$$

$$(Z_1 = H/K_s, Z_2 = G/K_s) \quad \ldots(51)$$

in which

$$\theta_1 = 1 + \frac{\beta' G + \sigma}{2(\beta' + 5)} \quad \ldots(52)$$

$$\theta_2 = 1 + \frac{\beta' H + \sigma}{2(\beta' + 5)} \quad \ldots(53)$$

$$\theta_3 = \frac{d_1}{8(\beta' + 5)}\left[\beta'(4 - 3d_1 + b_1) + 12\left(1 - d_1 + \frac{b_1}{2}\right)\right] \quad \ldots(54)$$

$$\theta_4 = \frac{d_1}{4}(1 - d_1 + b_1) \quad \ldots(55)$$

$$\theta_5 = \frac{\beta'\sigma^{G-4}(G - 3) - 6\sigma^{G-5}\sqrt{\sigma^m}}{2\beta'\sigma + 10\sqrt{\sigma^m}} \quad \ldots(56)$$

$$\theta_6 = \frac{\beta'\sigma^{H-4}(H - 3) - 6\sigma^{H-5}\sqrt{\sigma^m}}{2(\beta'\sigma + 10\sqrt{\sigma^m})} \quad \ldots(57)$$

$$\theta_7 = 16\bar{a}^2\sigma^2 + 24\bar{a}\bar{b} \quad \ldots(58)$$

$$\theta_8 = -6b_1d_1 + 2d_1^2 - 6b_1 - \tfrac{3}{8}b_1^2 . \quad \ldots(59)$$

Thus the velocity and pressure distributions are obtained.

Force on The Shell

The force exerted on the shell is given by

$$D_r = 4\pi a\mu U d_1\left(1 + R\frac{d_1}{4}\right) \quad \ldots(60)$$

where

$$d_1 = \frac{3(2 + \beta')(\varphi_3 h - \varphi_4 g)}{(\varphi_2\varphi_3 - \varphi_1\varphi_4)} \quad \ldots(61)$$

Particular Cases

Case 1

If the shell has a constant permeability K, then

$$d_1 = \frac{-3(2+\beta')(30\beta' K + 3\sigma + \beta'\sigma^2 - 3\sigma^3 - \beta'\sigma^5)}{2(\beta'+3-6K)(3\sigma^4+\beta'\sigma^5)-(2\beta'+3\beta' K+6K+6)(30\beta' K+3\sigma+\beta'\sigma^2)}. \quad \ldots(62)$$

The drag so determined by (60) with (62) agrees with the result obtained by Verma and Bhatt (1975).

Zeroth order of (60) with (62) agrees with Jones (1973).

Case 2

The drag on a porous sphere of variable permeability under the law of our study comes out to be

$$D = \frac{12\pi a\mu U(2+\beta')g}{\varphi_1}\left(1+\frac{3}{4}\frac{Rg(2+\beta')}{\varphi_1}\right). \quad \ldots(63)$$

Zeroth order of (63) agrees with the result determined by Verma and Vyas (1979).

In case the permeability K of porous sphere is constant, then the drag

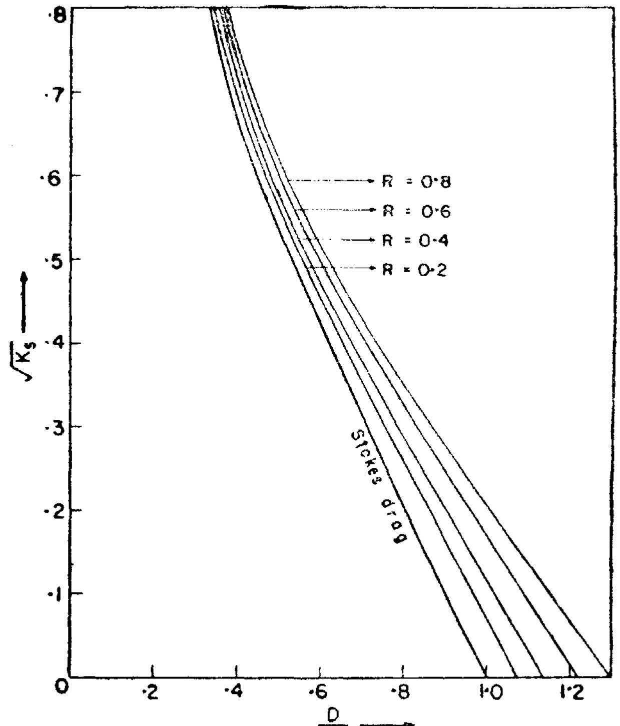

FIG. 1. D/D_s versus $\sqrt{K_s}$ for fixed $m = -2$ and $\alpha = 1$.

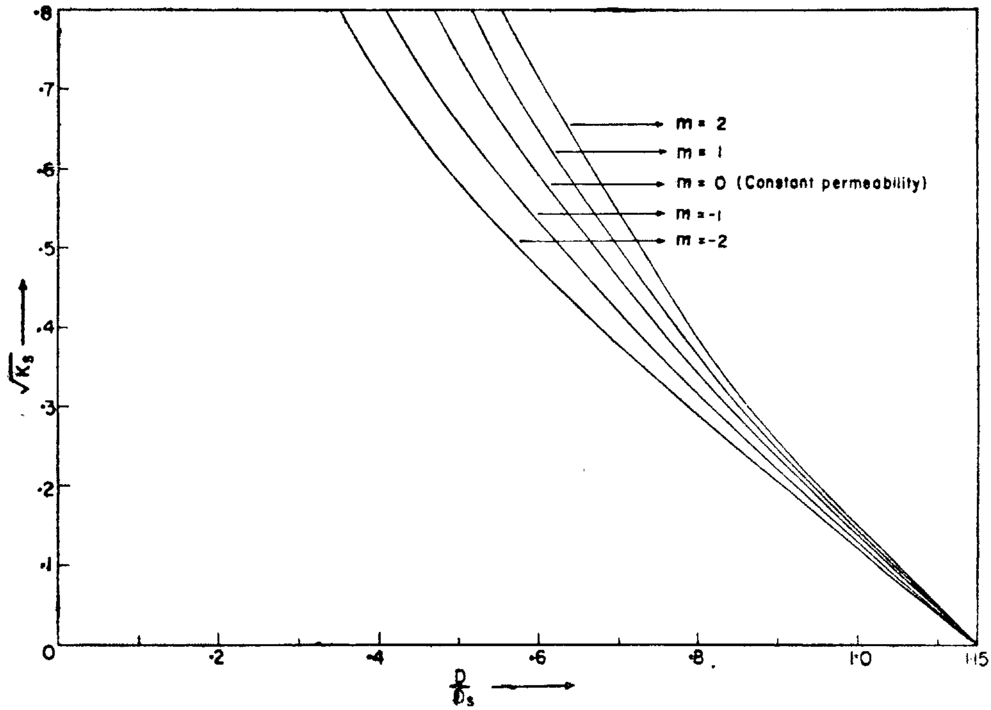

FIG. 2. D/D_s versus $\sqrt{K_s}$ for fixed $R = 0.4$ and $\alpha = 1$.

$$D = \frac{6\pi a\mu U(2 + \beta')}{(3K + 3 + \frac{3}{2}\beta'K + \beta')}\left[1 + \frac{3(2 + \beta')R}{8(\beta' + \frac{3}{2}\beta'K + 3K + 3)}\right] \quad \ldots(64)$$

which agrees with Verma and Bhatt (1957).

Zeroth order of (64) agrees with Jones (1973).

Now if the permeability $K \to 0$ i.e. $\beta' \to \infty$ (64) reduces to

$$D = 6\pi\mu aU\left(1 + \frac{3}{8}R\right),$$

which agrees with Proudman and Pearson (1957).

Numerical Discussions

The non-dimensional permeability K at a point within the sphere is given by $K = K_s r^m$,

where K_s is the non-dimensional surface permeability. Using (63) we have shown the variation of non-dimensional drag D/D_s, (D_s — Stokes' drag $= 6\pi\mu aU$) with $\sqrt{K_s}$ for $\alpha = 1$. In Fig. 1, the variation is shown for $m = -2$ and for different values of Reynolds number $R = 0.2$, 0.4, 0.6 and 0.8. It is inferred that for a given value of R drag D/D_s decreases as surface permeability K_s increases and also for fixed K_s drag D/D_s increases with increase of R. In Fig. 2, D/D_s is plotted against $\sqrt{K_s}$ for $R = 0.4$, $\alpha = 1$ and for different values of $m = -2, -1, 0, 1$ and 2. Here the drag D/D_s increases with m for fixed value of K_s, since K decreases with the increase of m for fixed value of K_s. Thus the drag increases or decreases with the decrease or increase of the permeability.

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