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AXISYMMETRIC BOUNDARY LAYER WITH SUCTION OVER A POROUS SPHERE

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An approximate solution has been obtained for the axisymmetric boundary layer with suction over a porous sphere. In the region of pressure decrease from the front stagnation point to the point of the maximum velocity the direct quadrature formula has been used. In the region of pressure rise the momentum and the kinetic energy integral equations have been numerically integrated with the aid of a singly infinite family of velocity profiles.

1. INTRODUCTION

Walz (1948) first developed a method based on the joint use of the momentum and the kinetic energy integral equations to study the two-dimensional laminar boundary layers along solid walls. Head (1961) put the momentum and the kinetic energy integral equations for two-dimensional boundary layers along porous walls in dimensionless forms and showed that results with sufficient accuracy could be achieved by the joint use of the two integral equations. Rott and Crabtree (1952) gave a direct quadrature formula for the calculation of the momentum thickness for axisymmetric boundary layers along solid walls.

In this paper the quadrature formula of Rott and Crabtree (1952) has been extended to the axisymmetric boundary layers with suction and has been used to obtain the momentum thickness for the boundary layer with suction over a sphere in the region of favourable pressure gradient ($0 \leqslant \bar{x} \leqslant \frac{1}{2}\pi$). The method suggested by Head (1961) has been extended to the investigation of the axisymmetric boundary layers with suction. The momentum and the kinetic energy integral equations for axisymmetric boundary layers have been used with the aid of Schlichting's (1949) profiles to obtain a step-by-step solution for the boundary layer with suction in the region of adverse pressure gradient ($\bar{x} > \frac{1}{2}\pi$) over a porous sphere.

2. EQUATIONS

The momentum and the kinetic energy integral equations for axisymmetric boundary layers with suction are given below.

(a) *Momentum Integral Equation*

The boundary layer equation for steady laminar incompressible flow past axisymmetric bodies is

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \qquad ...(1)$$

and the equation of continuity is

$$\frac{\partial (ur)}{\partial x} + \frac{\partial (vr)}{\partial y} = 0 \qquad ...(2)$$

Integrating eqn. (1) with respect to y from $y = 0$ to $y = \infty$ and using the equation of continuity (2), we obtain

$$\frac{d}{dx}\left(\frac{\theta^2}{\nu}\right) = \frac{2}{U}\,[l - (2 + H)\,\Lambda + \sigma - Z] \qquad ...(3)$$

where $\nu = \mu/\rho$, kinematic viscosity,

$U(x) =$ potential flow velocity

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$$

$$\theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

$$H = \frac{\delta^*}{\theta};\; l = \frac{\theta}{U}\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$\Lambda = \frac{\theta^2}{\nu}\frac{dU}{dx},\; Z = \frac{\theta^2}{\nu}\frac{U}{r}\frac{dr}{dx},$$

$$\sigma = \frac{v_s \theta}{\nu},\; \sigma < 0, \text{ suction}$$

$r(x) =$ radius of cross section.

Introducing the dimensionless quantities

$\bar{x} = x/a$, where a is a representative length,

$\bar{U} = U(x)/U_0$, where U_0 is the free stream velocity,

and $t^* = (\theta/a)^2 \,.\, U_0 a/\nu$,

the momentum integral equation in dimensionless form is

$$\frac{dt^*}{d\bar{x}} = \frac{2}{\bar{U}}\,[l - (2 + H)\,\Lambda + \sigma - Z]. \qquad ...(4)$$

(*b*) *Kinetic Energy Integral Equation*

Adding $\frac{u}{2r}\left[\frac{\partial(ur)}{\partial x} + \frac{\partial(vr)}{\partial y}\right]$ to the left-hand side of eqn. (1), multiplying through by u and integrating with respect to y from $y = 0$ to $y = \infty$ with the aid of eqn. (2) we get

$$\frac{d}{dx}\left(\frac{\epsilon^2}{\nu}\right) = \frac{2H_\epsilon}{U}[2D - H_\epsilon(3\Lambda + Z) + \sigma] \qquad \ldots(5)$$

where $$\epsilon = \int_0^\infty \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right)dy$$

$$D = \int_0^\infty \left(\frac{\theta}{U}\right)^2\left(\frac{\partial u}{\partial y}\right)^2 d\left(\frac{y}{\theta}\right)$$

$$H_\epsilon = \epsilon/\theta.$$

The direct variation of H_ϵ is given by

$$\frac{dH_\epsilon}{dx} = \frac{1}{\theta}\left[\frac{d\epsilon}{dx} - H_\epsilon\frac{d\theta}{dx}\right]$$

i.e. $$\frac{dH_\epsilon}{dx} = \frac{1}{Ut^*}[2D - H_\epsilon\{l - (H-1)\Lambda + \sigma\} + \sigma]. \qquad \ldots(6)$$

(*c*) *Wall Compatibility Condition*

At the surface of the body where $u = 0$, $v = v_s$ the boundary layer eqn. (1) reduces to

$$\nu\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = U\frac{dU}{dx} + v_s\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

i.e. $$m = -\Lambda + l\sigma \qquad \ldots(7)$$

where $$m = \frac{\theta^2}{u}\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}.$$

3. Family of Velocity Profiles

The one-parameter family of velocity profiles given by Schlichting (1949) for the approximate calculation of boundary layers with suction is

$$\frac{u}{U} = F_1(\eta) + KF_2(\eta) \qquad \ldots(8)$$

where $$\eta = \frac{y}{\delta(x)},\ F_1(\eta) = 1 - e^{-\eta}$$

$$F_2(\eta) = F_1(\eta) - \sin\frac{\eta\pi}{6}, \, 0 \leqslant \eta \leqslant 3$$
$$= F_1(\eta) - 1, \;\; \eta \geqslant 3$$

and $\delta(x)$ is a measure of the local boundary layer thickness.

The compatibility condition (7) takes the form

$$(K+1)\frac{\theta^2}{\delta^2} + \frac{\theta}{\delta}\,\sigma\left\{1 + K\left(1 - \frac{\pi}{6}\right)\right\} - \Lambda = 0. \quad \text{...(9)}$$

4. Direct Integration Formula

Thwaites (1949) constructed a simplified method and approximately represented the function in the numerator of the momentum integral equation for two dimensional boundary layers as

$$2\,[l - (2 + H)\,\Lambda + \sigma] = 0.45 - 6\Lambda + 1.28\sigma + 0.76\sigma^2. \quad \text{...(10)}$$

Following Rott and Crabtree (1952) and using the approximation eqn. (10) given by Thwaites (1949) the momentum integral eqn. (4) for the axisymmetric boundary layers with suction can be put in a simpler form as

$$\frac{d}{d\bar{x}}\,(\bar{U}^6\bar{r}^2t^*) = (0.45 + 1.28\sigma + 0.76\sigma^2)\,\bar{r}^2\bar{U}^5 \quad \text{...(11)}$$

where $\bar{r} = \dfrac{r}{a}, \; \bar{v}_s = \dfrac{v_s}{U_0}\left(\dfrac{U_0 a}{\nu}\right)^{1/2}$

$$\sigma = \frac{v_s\theta}{\nu} = (t^*)^{1/2}\,.\,\bar{v}_s.$$

Directly integrating eqn. (11) from $\bar{x} = 0$ to $\bar{x}$ for constant values of σ we have

$$t^* = \frac{0.45 + 1.28\sigma + 0.76\sigma^2}{\bar{r}^2\bar{U}^6}\int_0^{\bar{x}} \bar{r}^2\bar{U}^5 d\bar{x} \quad \text{...(12)}$$

Thwaites integration formula (12) would give sufficiently accurate values of t^* in the region of favourable pressure gradient but would diverge from the actual value in the region of adverse pressure gradient.

5. Porous Sphere

It is proposed to investigate the laminar incompressible boundary layer over a porous sphere.

The velocity distribution along the contour of the sphere is given by

$$U(x) = \tfrac{3}{2}\,U_0 \sin(x/R)$$

i.e. $\bar{U} = \frac{3}{2} \sin \bar{x}$

where $a = R$, radius of the sphere

$$\bar{x} = \frac{x}{R},$$

$$r(x) = R \sin \frac{x}{R}$$

i.e. $\bar{r} = \sin \bar{x}$

and $$\Lambda = t^* \frac{d\bar{U}}{d\bar{x}} = \frac{3}{2} t^* \cos \bar{x}$$

Equations (4) and (6) and the compatibility condition (9) become

$$\frac{dt^*}{d\bar{x}} = f(\bar{x}, t^*, H_\epsilon) \qquad \text{...(13)}$$

where

$$f(\bar{x}, t^*, H_\epsilon) = \frac{4}{3 \sin \bar{x}} [l - \tfrac{3}{2} t^*(H + 3 \cos \bar{x} + \sigma]$$

$$\frac{dH_\epsilon}{d\bar{x}} = g(\bar{x}, t^*, H_\epsilon) \qquad \text{...(14)}$$

where

$$g(\bar{x}, t^*, H_\epsilon) = \frac{2}{3t^* \sin \bar{x}} [\, 2D - H_\epsilon \{l - \tfrac{3}{2} t^*(H - 1) \cos \bar{x} + \sigma\} + \sigma]$$

and

$$(K + 1) \frac{\theta^2}{\delta^2} + \sigma \frac{\theta}{\delta} \left\{1 + K\left(1 - \frac{\pi}{6}\right)\right\} - \tfrac{3}{2} t^* \cos \bar{x} = 0 \qquad \text{...(15)}$$

(a) *Direct Integration*

For the porous sphere eqn. (12) becomes

$$t^* = \frac{2}{3} \left[\frac{0.45 + 1.28\sigma + 0.76\sigma^2}{\sin^8 \bar{x}}\right] \int_0^{\bar{x}} \sin^7 \bar{x} \, d\bar{x}$$

i.e. $$t^* = \frac{2}{3} \left[\frac{0.45 + 1.28\sigma + 0.76\sigma^2}{\sin^8 \bar{x}}\right] \left[\frac{16}{35} - \frac{\cos \bar{x} \sin^6 \bar{x}}{7} - \frac{6}{35} \cos \bar{x} \sin^4 \bar{x} - \frac{8}{35} \cos \bar{x} \sin^2 \bar{x}\right]. \qquad \text{...(16)}$$

At the stagnation point $\bar{x} = 0$, eqn. (16) takes the indeterminate form $\frac{0}{0}$ and the value of t^* is obtained by the simple process of taking the limit.

At any other point $0 < x \leqslant \frac{1}{2}\pi$ the value of t^* is computed by eqn. (16) and with the known value of t^* the value of the profile parameter K is obtained by a numerical solution of the compatibility condition (15). The value of K gives the other corresponding parameters H_ε, H, l and D (Mishra and Choudhary 1972).

(b) *Solution of the Momentum and the Kinetic Energy Integral Equations*

After having obtained the values at $\bar{x} = \frac{1}{2}\pi$ the momentum integral eqn. (13) only has been solved for a few steps by Runge-Kutta method (Scarborough 1956) with the satisfaction of compatibility condition (15).

After a few steps the momentum and the kinetic energy integral equations both have been solved by Runge-Kutta method. Having obtained the values at five initial points by Runge-Kutta method repeated ascending differences upto the fourth are computed for the functions f and g. The increments in t^* and H_ϵ over a step-length are calculated by Adams method (Scarborough 1956) using a quadrature formula. Taking the step-length $\triangle \bar{x} = 0.01$ the calculations have been made step-by-step up to the point of separation given by $l = 0$.

6. Results

Calculations have been made for three different constant values of $\sigma = 0$, -0.2, -0.4. The results of calculations are shown in Figs. 1 and 2. With an increase in the rate of suction the boundary layer thickness decreases (Fig. 1) and the point of separation shifts downstream.

For $\sigma = 0$ the problem reduces to the solid boundary problem and the point of separation obtained by the present method is $\bar{x} = 1.979$ which agrees well with the

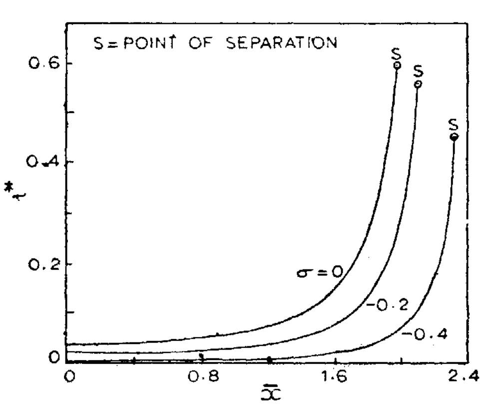

FIG. 1. Variation of t^* against $\bar{x}$ for different values of σ.

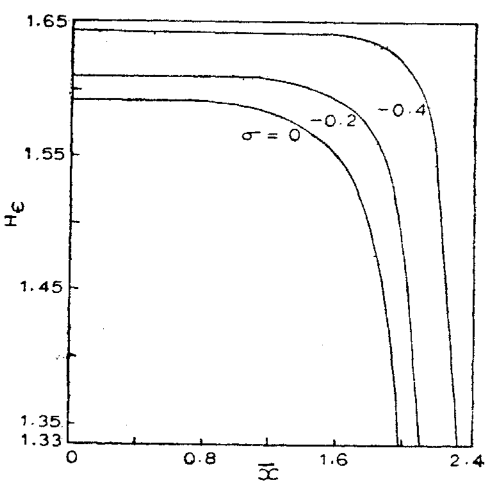

FIG. 2. Variation of H_ϵ against $\bar{x}$ for different values of σ.

point of separation at $\bar{x} = 1.913$ given by the exact method of solution (Schlichting 1968). It is expected that the results with suction would be sufficiently accurate.

Point of separation on a porous sphere

Suction parameter	Point of separation	
$\sigma = t^{*1/2}\bar{v}_s$	$\bar{x}_s$	ϕ_s
0.0	1.979	113°24′
−0.2	2.100	120°28′
−0.4	2.320	130°14′

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